APPRIXIMATE SHORTEST DISTANCES AMONG SMOOTH OBSTACLES IN 3D

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ABSTRACT. We consider the classic all-pairs-shortest-paths (APSP) problem in a three-dimensional environment where paths have to avoid a set of smooth obstacles whose surfaces are represented by discrete point sets with \( n \) sample points in total. We show that if the point sets represent \( \varepsilon \)-samples of the underlying surfaces, \( (1 \pm O(\sqrt{\varepsilon})) \)-approximations of the distances between all pairs of sample points can be computed in \( O(n^{5/2} \log^2 n) \) time.

1 Introduction

Computing shortest distances between pairs of points is one of the classic problems in Computational Geometry. The problem is well-understood in (geometric) graphs and in the Euclidean plane [11]. Computing exact shortest paths among obstacles whose exact polyhedral surfaces are known as part of the input in three-dimensional Euclidean space, however, was shown to be \( \mathcal{NP} \)-hard [7]. Subsequently, authors have considered special cases such as exact distances on a convex polyhedron [16] or approximate distances on a general, possible weighted polyhedron [2], see also the survey by Bose et al. [6].

We consider a set of smooth obstacles in \( \mathbb{R}^3 \) given as an \( \varepsilon \)-sample, i.e., a point set on the union of the obstacles’ boundaries locally dense enough to faithfully capture curvature and folding. As usual, \( \varepsilon \) is a sampling parameter unknown to the algorithm [4, 14]. In line with previous approaches (see [15] and the references therein), we assume that \( \varepsilon \) is upper-bounded by a constant \( 1 > \varepsilon_0 > 0 \) which only depends on the algorithm but neither on the input size nor on the curvature or folding of the underlying surface. We obtain the following result:

**Theorem 1.** There is a global and shape-independent constant \( \varepsilon_0 > 0 \) such that it holds for \( \varepsilon \leq \varepsilon_0 \): Given an \( \varepsilon \)-sample \( S \) of a set of smooth obstacles in \( \mathbb{R}^3 \), we can compute \( (1 \pm O(\sqrt{\varepsilon})) \)-approximations of all \( \binom{n}{2} \) distances in \( O(n^{5/2} \log^2 n) \) time, where \( n := |S| \).

In general, shortest paths among obstacles alternate between geodesic subpaths on obstacles and straight-line segments in free space. The standard approach to computing such free-space geodesics would be to compute both geodesic distances and visibility edges between each pair of points, model these distances by a weighted graph \( G \) with vertices...
corresponding to the sample points on the obstacles, and then to combine the results using an all-pairs shortest path algorithm on $G$. Due to the complexity of visibility maps in three-dimensional space, this approach would lead to an at least cubic runtime. We will alleviate this problem by simultaneously restricting the degree of $G$ and locally bounding the length of the edges.

**Related Work**  
A free-space geodesic is modeled by two types of edges in $G$ that correspond to either geodesics on obstacles or straight-line segments in free space. For the geodesics on obstacles, we note that exact shortest path computations on general polyhedra are considered complex and challenging. We refer to recent surveys [3, 6, 11] for a detailed discussion and focus on two approximation algorithms: the best algorithm currently known for weighted polyhedra is the algorithm by Aleksandrov et al. [3], while the algorithm by Scheffer and Vahrenhold [14] works on (unweighted) 2-manifolds in $\mathbb{R}^3$. The efficiency of the algorithm by Aleksandrov et al. depends on the triangles obeying a “fatness” condition. In general, however, the aspect ratio and, hence, the runtime can be arbitrarily large. Both algorithms result in a shortest-path graph of quadratic size. Exploring this structure in combination with a visibility graph (see below) as part of a shortest-path algorithm leads to at least cubic runtime.

The second type of edges of a free-space geodesic corresponds to straight-line segments connecting points on obstacles by crossing the free space; these *bridge edges* are obtained by computing visibility information between points on the obstacles. Despite recent advances in algorithms for “realistic terrains” [12], the complexity of the visibility map of a three-dimensional surface is quadratic in the worst case—see [12] and the references therein. As discussed above, a visibility map of quadratic size leads to a cubic overall running time. A subquadratic complexity currently can be obtained only under standard assumptions about the fatness of the triangles and the (bounded) ratio of shortest and longest edges; in particular the latter assumption is infeasible in the case we are considering.

**2 Outline of the Algorithm**

From a high-level perspective, our algorithm proceeds by first constructing a weighted graph $G := (S^{\text{sub}}, E_{\text{loc}} \cup E_{\text{bri}})$ on a subset $S^{\text{sub}} \subseteq S$ of sample points where the edges in $E_{\text{loc}}$, called *local edges*, represent approximate free-space geodesic distances between points on $\Gamma$ and the edges in $E_{\text{bri}}$ represent straight-line segments avoiding $\Gamma$; in either case, the weight of an edge denotes the length of the respective connection. The algorithm then approximates the free-space geodesic distances $L^*_\Gamma(s_1, s_2)$ for all $s_1, s_2 \in S^{\text{sub}}$ by computing exact all-pairs shortest paths in $G$ and extends these results to compute approximate distances $L(\cdot, \cdot)$ between all points in $S$. The algorithm is given below and will be discussed in the following.

For clarity of exposition, we will use the term *approximation error* to denote the maximum of $\frac{L^*_\Gamma(s_1, s_2)}{L(s_1, s_2)}$ and $\frac{L(s_1, s_2)}{L^*_\Gamma(s_1, s_2)}$ maximized over $s_1, s_2 \in S$.\(^1\)

\(^1\)To improve readability, we use fraktur letters to denote points in the input point set $S$; for these points, we only use the property that $S$ is an $\varepsilon$-sample. Regular letters are used to denote points that are part of the coarsened subset $S^{\text{sub}} \subseteq S$, thus have additional properties, or points on the manifold not in the sample.
Algorithm 1 Approximating Geodesic Distances.

1: function APX3DGEODESICDISTANCES($S$)  
2:  \( \psi(\cdot) \leftarrow \text{CONTROLFUNCTION}(S); \) \hspace{1em} \( \triangleright \) Setup approximation using Lemma 1 \([9, 14]\) \ldots
3:  \( \text{afs}(\cdot) \leftarrow \text{APXLOCALFEATURESIZE}(S); \) \( \triangleright \ldots \) and algorithm by Aichholzer et al. \([1]\)
4:  \( \delta \leftarrow \max_{s \in S} \frac{\psi(s)}{\text{afs}(s)}; \) \( \triangleright \) Lower-bound local feature size
5:  \( S_{\text{sub}} \leftarrow \text{COARSENSAMPLE}(S, \delta, \text{afs}(\cdot)); \) \( \triangleright \) Use Algorithm 2
6:  \( E_{\text{loc}} \leftarrow \text{COMPUTELOCALEDGES}(S_{\text{sub}}, \delta, \text{afs}(\cdot)); \) \( \triangleright \) Use Algorithm 3
7:  \( E_{\text{bri}} \leftarrow \text{COMPUTEBRIDGEEDGES}(S_{\text{sub}}, S, \delta, \text{afs}(\cdot)); \) \( \triangleright \) Use Algorithm 5
8:  \( G \leftarrow (S_{\text{sub}}, E_{\text{loc}} \cup E_{\text{bri}}); \) \( \triangleright \) Assemble graph \( G \)
9:  return APXDISTANCESFROMGRAPH($S$, $G$); \( \triangleright \) Expand result from $S_{\text{sub}}$ to $S$

As sketched above, the main challenge in obtaining an algorithm with subcubic running time lies in working with a shortest-path graph with bounded or at least sublinear degree. As we will detail below, we can obtain such a graph by first avoiding to compute approximate free-space geodesic distances between all pairs of points. Instead, we will compute such distances only for points within a locally bounded distance of each other. These distances will be represented by the set \( E_{\text{loc}} \) of local edges connecting points in a carefully coarsened subsample \( S_{\text{sub}} \subseteq S \). We then compute a set \( E_{\text{bri}} \) of bridge edges such that \( E_{\text{bri}} \) is a superset of the visibility graph of \( S_{\text{sub}} \) w.r.t. (the sample points of) \( \Gamma \). In both cases, we can simultaneously guarantee a sublinear node degree and a small approximation error.

In the remainder of this section, we consider both types of edges in turn and show how to efficiently compute them. We then discuss how to combine local and bridge edges into a graph \( G \) that is sparse enough to be traversed efficiently. Finally, we analyze the resulting algorithm w.r.t. to its running time and approximation quality.

2.1 Computing Local Edges

The situation we are facing is similar to the construction of spanner graphs approximating the full Euclidean graph. One way of constructing a spanner graph is by means of the well-separated pair decomposition \([10]\). Informally, this approach connects (representative points of) clusters that are “far away” from each other, where the notion of “far away” depends on the radius of the clusters. Doing so, the intra-cluster distances are approximated by the length of the edge connecting the representatives.

We proceed along similar lines: we consider only edges between points that are at bounded distance from each other where the notion of “bounded” depends on the sampling density and the curvature and folding of \( \Gamma \) at the points in question. This allows us to relate Euclidean and geodesic distances and thus to upper-bound the approximation quality of the geodesic distances computed. A naive implementation of this approach, however, might lead to a linear number of edges per point. To avoid such situations, we need to ensure that only few “short” edges are constructed per point; this will be done by applying a standard preprocessing step in which the sample is coarsened appropriately without affecting the sampling quality \([14]\).
In contrast to the construction of the spanner graph, the construction steps sketched do not guarantee a linear number of edges to be sufficient to obtain a good approximation error. What we can show, however, is that by considering only edges for points whose Euclidean distance is upper-bounded as sketched above, we obtain a graph with $O(n^{3/2})$ edges which still results in the desired approximation quality.

### 2.1.1 Characterizing and Approximating the Sampling Density

To measure the density of a point sample, it is customary to consider the **local feature size** $lfs(\cdot)$ that is defined as the distance function to the medial axis of $\Gamma$ [4]. To formalize notation, a discrete subset $S$ of $\Gamma \subset \mathbb{R}^3$ is an **$\varepsilon$-sample** of $\Gamma$ if for every point $x \in \Gamma$ there is a sample point $s \in S$ such that its distance $|xs|$ to $s$ is upper-bounded by $\varepsilon \cdot lfs(x)$. The local feature size $lfs(\cdot)$ is $c$-Lipschitz for $c = 1$, i.e., $lfs(x) \leq lfs(y) + c|xy| = lfs(y) + |xy|$, $x, y \in \mathbb{R}^3$.

In [14], we show that $|x_1x_2|$ for $x_1, x_2 \in \Gamma$ is an $(1 + O(\varepsilon))$-approximation of the geodesic distance $L_{\Gamma}(x_1, x_2)$ on $\Gamma$ between $x_1$ and $x_2$ if $|x_1x_2| \leq \sqrt{\varepsilon} \cdot \min\{lfs(x_1), lfs(x_2)\}$. Put differently, if points are “close enough” to each other, their Euclidean distance approximates their geodesic distance. While we use this upper bound in [14] to prove the approximation quality of the geodesic distances computed, the discussion above suggests that our algorithm needs to actually evaluate these expressions to compute the radii of the locally bounded neighborhoods mentioned above and thus to be able to exclude points at larger distance from comparison. Unfortunately, neither $\varepsilon$ nor $lfs(\cdot)$ can be computed exactly as $\Gamma$ is unknown.

What we can do, however, is to compute an approximate upper bound $afs(\cdot)$, the **approximate local feature size**, for $lfs(\cdot)$ such that $lfs(s) / afs(s)$ for all $s \in S$. Furthermore, we compute a so-called control function $\psi(\cdot)$ such that $\psi(s) \geq O(\varepsilon) \cdot lfs(s)$ for all $s \in S$. Finally, we compute an approximate lower bound $\delta := \max_{s \in S} (\psi(s) / afs(s))$ for $\varepsilon$. Following Aichholzer et al. [1], we consider the distance from $s \in S$ to the closest pole to approximate the local feature size.

**Definition 1** ([4]). The **poles** of some $s \in S$ are the two vertices of the Voronoi cell $Vor_S(s)$ of $s$ in the Voronoi diagram of $S$ which are farthest from $s$, one on either side of $\Gamma$ (note that $\Gamma$ is the boundary of a 2-manifold, hence the inside and outside are well-defined). A pole $p_s$ of $s$ is called an **outer** pole if it lies outside $\Gamma$, and an **inner** pole otherwise.

Aichholzer et al. observed that this distance is the desired approximate upper bound $afs(s)$ for the local feature size $lfs(s)$, i.e., that $lfs(s) \leq 1.2802 \cdot afs(s)$ holds. Obviously, we can compute the (Voronoi diagram and the) poles and, hence, $afs(\cdot)$ in quadratic time. While the algorithm only needs to know the values of $afs(\cdot)$ for all points in $S$, the analysis will use a version of this function lifted to $\Gamma \subset \mathbb{R}^3$. For this analysis, we can assume that we can extend the domain of $afs(\cdot)$ to $\Gamma$—if needed, pointwise — such that $lfs(x) \leq 1.2802 \cdot afs(x)$ holds for all $x \in \Gamma$ (note that $lfs(\cdot)$ is defined for each point $x \in \Gamma$).

**Observation 1.** $afs(\cdot)$ is $1$-Lipschitz and can be computed in $O(n^2)$ time.

We then obtain the approximate lower bound $\delta$ for $\varepsilon$ using a **control function**: 
Lemma 1 ([14]). We can compute in time $O(n^2)$ a control function $\psi : S \rightarrow \mathbb{R}^+$ such that: (1) $\forall s \in S : \psi(s) \leq 1.19 \cdot \varepsilon \cdot ||s||$, (2) $\forall s \in S : \forall x \in \text{Vor}(s) \cap \Gamma : |xs| \leq \psi(s)$, and (3) $\psi$ is $\frac{1}{15}$-Lipschitz.

In line with the above argumentation, for $s_1, s_2 \in S$ we show $|s_1s_2| \leq (1 + O(\sqrt{\varepsilon})) \cdot L^*_1(s_1, s_2)$ if $|s_1s_2| \leq \frac{1}{3} \cdot \sqrt{\delta} \cdot \min \{afs(s_1), afs(s_2)\}$ holds, i.e., the above approximation scheme still yields meaningful results for sample points $s_1, s_2 \in S$ “close enough” to each other.

In our previous work [15], we proved:

Lemma 2 ([15, Lemma 23]). There is a global, shape-independent constant $\varepsilon_0$ such that for all $\varepsilon \leq \varepsilon_0$, the approximate lower bound $\delta$ for $\varepsilon$ satisfies $\delta \in O(\sqrt{n})$.

2.1.2 Coarsening the Sample

It remains to discuss how to avoid high-degree nodes in the distance graph, or, equivalently, to avoid connecting points to “too many” other points that fulfill the above distance criterion. For this, we use the control function implied by Lemma 1 to compute a coarsened subsample $S^{\text{sub}} \subseteq S$ in which the following two conditions hold:

1. For each point $x \in \Gamma$, there is a sample point $s \in S^{\text{sub}}$, such that $|xs| \leq O(\delta) \cdot afs(s)$.
2. For any two sample points $s \neq s' \in S^{\text{sub}}$, $|ss'| \geq O(\delta) \cdot afs(s)$ holds.

Algorithm 2 Compute a coarsened subsample $S^{\text{sub}} \subseteq S$ (see [9, 14]).

1: function COARSENsample($S$, $\delta$, $afs(\cdot)$)
2: \hspace{1em} $S^{\text{sub}} \leftarrow \emptyset$; $\beta \leftarrow 0.1$; \hspace{1em} $\triangleright$ Fix constant in “big-oh”-notation for later analysis
3: \hspace{1em} while $S \neq \emptyset$ do
4: \hspace{2em} $s \leftarrow$ arbitrary point in $S$;
5: \hspace{2em} $S^{\text{sub}} \leftarrow S^{\text{sub}} \cup \{s\}$;
6: \hspace{2em} $S \leftarrow S \setminus B_{\beta \cdot \delta \cdot afs(s)}(s)$; \hspace{1em} $\triangleright B_r(x)$: ball with radius $r$ centered at $x$
7: \hspace{1em} return $S^{\text{sub}}$;

2.1.3 Intermediate Summary: Computing Local Edges

Summarizing the above discussion, we first coarsen the subsample using Algorithm 2 and then construct local edges between all points that are close enough—see Algorithm 3.

Lemma 3. Let $S$ be an $\varepsilon$-sample and $E_{\text{loc}}$ the set of local edges computed for a coarsened subsample $S^{\text{sub}} \subseteq S$ according to Algorithm 2 and 3. Then, the following properties hold:

(LE1) The length of a local edge is a $(1 \pm O(\sqrt{\varepsilon}))$-approximation of the geodesic distance of its endpoints. More precisely, for all $s_1, s_2 \in S^{\text{sub}}$ such that $|s_1s_2| \leq \frac{1}{3} \cdot \sqrt{\delta} \cdot \min \{afs(s_1), afs(s_2)\}$ holds, we have $L^*_1(s_1, s_2) \leq (1 + O(\delta)) \cdot |s_1s_2| \leq (1 + O(\sqrt{\varepsilon})) \cdot |s_1s_2|$.
Proof. (LE1) (To enhance the accessibility of this part of the paper, we defer the rather technical proof of this property to Section 3.1.1 (Lemma 16)).

(LE2) Fix a point \( s \in S_{\text{sub}} \) and let \( x \) be any point in \( \text{Vor}_{S_{\text{sub}}}(s) \). By Lemma 4 there is some \( s' \in S_{\text{sub}} \) such that \( |xs'| \leq 1.17 \cdot \delta \cdot \text{afs}(s') \). Since \( \text{afs}(\cdot) \) is Lipschitz, it follows that \( |xs| \leq 1.2 \cdot \delta \cdot \text{afs}(x) \). Since \( x \in \text{Vor}_{S_{\text{sub}}}(s) \), we have \( |xs| \leq |xs'| \leq 1.2 \cdot \delta \cdot \text{afs}(x) \). Again, since \( \text{afs}(\cdot) \) is Lipschitz, we conclude that \( |xs| \leq \Theta(\delta) \cdot \text{afs}(s) \). Now, consider any local edge \( (s_1, s_2) \in S_{\text{sub}} \times S_{\text{sub}} \). By construction, the maximum length \( \lambda_{\text{max}} \) of any such edge is \( \lambda_{\text{max}} := \frac{1}{3} \cdot \sqrt{\delta} \cdot \min \{ \text{afs}(s_1), \text{afs}(s_2) \} \). Even if \( \text{afs}(s_1) = \max \{ \text{afs}(s_1), \text{afs}(s_2) \} \), the fact that \( \text{afs}(\cdot) \) is Lipschitz, together with the small distance of \( s_1 \) and \( s_2 \), implies that \( \lambda_{\text{max}} \geq \text{afs}(s_1) \cdot \Theta(\sqrt{\delta}) \). Thus, \( \lambda_{\text{max}} \cdot \Theta(\sqrt{\delta}) \geq |xs| \). The same argument applies to \( s_2 \).

(LE3) As \( \text{afs}(\cdot) \) is 1-Lipschitz, we can show \( \text{afs}(s') \geq (1 - \frac{1}{4} \cdot \delta) \cdot \text{afs}(s) \geq \frac{1}{2} \cdot \text{afs}(s) \) for \( s' \in B_{\frac{1}{4} \cdot \sqrt{\delta} \cdot \text{afs}(s)}(s) \). Algorithm 2 guarantees \( |ss'| \geq 0.1 \cdot \delta \cdot \text{afs}(s) \) for all \( s, s' \in S_{\text{sub}} \), \( s \neq s' \). A standard packing argument yields \( |B_{\frac{1}{4} \cdot \sqrt{\delta} \cdot \text{afs}(s)}(s) \cap S_{\text{sub}}| \leq \frac{1}{5} \). As all sample points connected to \( s \) by local edges lie inside \( B_{\frac{1}{4} \cdot \sqrt{\delta} \cdot \text{afs}(s)}(s) \), we can upper-bound the number of local edges incident to \( s \) by \( \frac{1}{5} \in \mathcal{O}(\sqrt{n}) \); see Lemma 2.

\[ \Box \]

Lemma 4. For each \( x \in \Gamma \), there is an \( s' \in S_{\text{sub}} \) with \( |xs'| \leq 1.17 \cdot \delta \cdot \text{afs}(s') \).

Proof. Let \( q \in S \) be the sample point closest to \( x \). We distinguish between two cases:

1. \( q \in S_{\text{sub}} \): We define \( s' := q \). By definition, \( \delta = \max_{s \in S} \frac{\psi(s)}{\text{afs}(s)} \geq \frac{\psi(s')}{\text{afs}(s')} \). Since, by Lemma 1, \( |xs'| \leq \psi(s') \), we have \( |xs'| \leq \frac{\psi(s')}{\text{afs}(s')} \cdot \text{afs}(s') \leq \delta \cdot \text{afs}(s') \).

2. \( q \in S \setminus S_{\text{sub}} \): Define \( s' \in S_{\text{sub}} \) to be the sample point that was processed by Algorithm 2 when \( q \) was excluded from further consideration (Line 6). With \( \beta = 0.1 \), this implies
that $|s'q| \leq 0.1 \cdot \delta \cdot afs(s')$. As $afs(\cdot)$ is 1-Lipschitz, we get $afs(q) \leq (1 + 0.1 \cdot \delta) \cdot afs(s')$. Since $q$ is the sample point closest to $x$, we have $|xq| \leq \delta \cdot afs(q)$ (see above). The triangle inequality implies then $|xs'| \leq |xq| + |qs'| \leq \delta \cdot afs(q) + 0.1 \cdot \delta \cdot afs(s') \leq (1 + 0.1 \cdot \delta) \cdot \delta \cdot afs(s') + 0.1 \cdot \delta \cdot afs(s') \leq 1.17 \cdot \delta \cdot afs(s')$ (since $\delta^2 < \delta$).

2.2 Computing Bridge Edges

The second type of edges used in our construction is the set $E_{br}$ of bridge edges. While we would ideally compute the visibility graph of $S_{sub}$ w.r.t. $\Gamma$, we cannot do so as the exact geometry of $\Gamma$ is unknown. We thus compute $E_{br}$ as a superset of the edges in the visibility graph making sure that the additional edges that may intersect the interior of $\Gamma$ do so not too deep; this will enable us to bound the approximation error.

2.2.1 Computing Approximate Visibility Information

As the exact nature of $\Gamma$ is unknown, we cannot compute the visibility map of $s' \in S_{sub}$ w.r.t. $\Gamma$. Neither can we use a polyhedral reconstruction of $\Gamma$ as the visibility map of $s'$ w.r.t. such a reconstruction may have quadratic complexity [12, Sec. 2.1]. To circumvent this problem, we refrain from reconstructing $\Gamma$ at all. Instead, we discretize $\Gamma$ by a set of carefully constructed skewed cubes corresponding to all points in $S$ and compute the visibility map of $s'$ w.r.t. these skewed cubes. The skew of these cubes will depend on their position relative to the point $s' \in S_{sub}$ under consideration, hence reducing the combinatorial complexity of the visibility map—see below. To emphasize this dependency of a cube’s skew, we will refer to such a cube as an $s'$-skewed cube.

For this approach to be effective, we require that the visibility information obtained in this way approximates the true visibility information. First, we require that the $s'$-skewed cubes indeed cover $\Gamma$ (recall that $\Gamma$ is the boundary of a manifold $\Sigma$, not $\Sigma$ itself) such that no obstacles are ignored or holes appear, and, second, we require that the $s'$-skewed cubes do not cover too much of the space outside $\Gamma$ in the sense that they block visibility rays that $\Gamma$ does not block. To fulfill these requirements, we compute, for each $s'' \in S_{sub}$, an $s'$-skewed cube that is centered at $s''$ and contains a ball of radius $2 \cdot \delta \cdot afs(s'')$ (intuitively, this means that the cubes are large enough to overlap with “neighboring” cubes and thus cover $\Gamma$). We then push all $s'$-skewed cubes towards the interior of $\Sigma$ such that they do not protrude from $\Sigma$ and thus block visibility rays that $\Sigma$ would not block. Based upon Amenta and Bern’s [4] observation that for any point $s \in S$ the vector from $s$ towards one of its poles approximates the respective surface normals in $s$, we push the $s'$-skewed cubes along the vector from $s$ towards its inner pole—see Algorithm 4.

For technical reasons, when computing the visibility information of a point $s' \in S_{sub}$, we cover the space with a constant number of pyramids with apex at $s'$ and compute the visibility information for each pyramid separately. We use $\Pi_{s'}$ to refer to this set of pyramids.

We are now ready to formalize the skew of the cubes. After we have identified all $s'$-skewed cubes intersecting a pyramid $\pi \in \Pi_{s'}$ with apex $s'$ (this takes linear time per
Algorithm 4 Compute centers of the cubes pushed towards the interior of the manifold $\Sigma$.
1: function ComputeCubeCenters($S, \delta, afs(\cdot)$)
2:    for all $s \in S$ do
3:         $p_s \leftarrow$ inner pole of $s$;
4:         $s^1 \leftarrow s + 15 \cdot \delta \cdot afs(\cdot)s \cdot \frac{\bar{sp}_s}{|\bar{sp}_s|}$; $\triangleright$ The constant 15 will be used in the proof of (BE1)
5:         $S^1 \leftarrow S^1 \cup \{s^1\};$
6:    return $S^1;$

Figure 1: Left: Construction of an $s'$-skewed cube $c := c_{s,\pi} \in \mathbb{C}(\pi, s')$ inside a pyramid. The base of $c$ is a cube of side length $2 \cdot \delta \cdot lfs(s)$ axis-aligned with the base of the pyramid $\pi$, i.e., $B_{2\cdot lfs(s)}(s) \subset c$. The sides of $c$ are slanted outwards from the back face by the same angle as the aperture of $\pi$. Finally, $c$ is pushed into the interior of the solid $\Sigma$ bounded by $\Gamma$ by translating $s$ into the direction of $p_s$ by $15 \cdot \delta \cdot afs(s)$ where $p_s$ is the inner pole point corresponding to $s$. Middle: Cross-section of the pyramid $\pi$ and some $s'$-skewed cubes during the sweep. Right: Projection of some $s'$-skewed cubes inside $\pi$.

each of the $O(1)$ pyramids), we will perform a top-down sweep over all $s'$-skewed cubes crossing $\pi$ and all sample points in $S_{\text{sub}}$ inside the cone and maintain only the cross-section of the sweeping plane with the scene. Whenever we encounter a sample point, we will locate this point in the cross-section and check its visibility from $s'$. At this point, we need the skew of the $s'$-skewed cubes to ensure that maintaining the cross-section is not too costly, i.e., that the geometry of the $s'$-skewed cubes does not induce too many events where the combinatorial nature of the cross-section changes. This is guaranteed by the following definition.

Definition 2. Given a pyramid $\pi \in \Pi_{s'}$ and a sample point $s \in S \setminus \{s'\}$, the $s'$-skewed cube $c := c_{s,\pi}$ of $s$ has the following properties—see Figure 1:

1. The front and back face of the $s'$-skewed cube $c$ are parallel to the base of $\pi$.
2. The sides of $c$ are slanted outwards from the back face by the same angle as $\pi$’s aperture.$^2$
3. The $s'$-skewed cube $c$ is centered at $s^1 := s + 15 \cdot \delta \cdot afs(s) \cdot \frac{\bar{sp}_s}{|\bar{sp}_s|}$ where $p_s$ is the inner pole point corresponding to $s$.

$^2$Here we need that we are working with a constant number of pyramids per point.
An important property of the $s'$-skewed cubes is that they are properly sandwiched between two balls whose radii depend on the (approximate) feature size:

**Lemma 5.** For each sample point $s \in S \setminus \{s'\}$, the $s'$-skewed cube $c := c_{s,π}$ constructed as described above, the following inclusion property holds: $B_{2.δ \cdot afs(s)}(s') \subset c \subset B_{3.82.δ \cdot afs(s)}(s')$.

**Proof.** The inclusion $B_{2.δ \cdot afs(s)}(s') \subset c$ directly follows from the constructive definition of $c$, see Figure 1. To show the second inclusion, we observe that the maximal distance between $s'$ and a point $x \in c$ is realized by a corner $b$ of the stretched front face of $c$. We can upper-bound $|s'b| \leq (2 \cdot \sqrt{2} \cdot \sin(10°) + \sqrt{2}) \cdot 2 \cdot δafs(s) \leq 3.82 \cdot δ \cdot afs(s)$. \hfill \(\square\)

For fixed $s' \in S_{\text{sub}}$ and $π \in Π_{s'}$, we denote the set of all $s'$-skewed cubes intersecting $π$ by $C(π,s')$. We define the visible neighborhood $V(π,s')$ of $s'$ w.r.t. $π$ as the union of all points from $S_{\text{sub}} \cap π$ that are visible from $s'$ w.r.t. $C(π,s')$.

**Lemma 6.** For fixed $s' \in S_{\text{sub}}$ and $π$, we can compute $V(π,s')$ in $O(n \log^2 n)$ time.

**Proof.** For ease of exposition, we assume w.l.o.g. that the base of the pyramid $π$ is aligned with the $yz$-plane—see Figure 1 (middle). We perform a standard space-sweep in which we process the points and the $s'$-skewed cubes' front faces in radial order from the top to the bottom face of the pyramid $π$. The important fact to note is that, by construction, the visible silhouette of the set of these $s'$-skewed cubes is exactly the projection of the set of their front faces onto the base of the pyramid—see Figure 1 (right). The sweep-line structure maintained by the algorithm is a segment tree $T$ over the $x$-coordinates of the projections of these front faces. Whenever we encounter the top edge of a front face $f$, we add $f$ to the set of obstacles currently active but inserting its $x$-interval into the sweep-line structure. At each node of $T$ whose extent is covered by $f$, we insert $f$ into a list of faces sorted by their distance to $s'$. Analogously, we remove a face from $T$ once we encounter its bottom edge. Because of the way the $s'$-skewed cubes have been constructed, i.e., because the aperture of the pyramid $π$ and the slanting angle of the $s'$-skewed cube coincide, the intersection of the sweeping plane and the $s'$-skewed cubes changes only at the top and bottom edges of the $s'$-skewed cubes. Whenever we encounter a sample point $s$, we query $T$ with the $x$-coordinate of $s$. At each node of $T$ visited, we check the sorted list of faces to see whether there is any face currently stored in $T$ that blocks $s$ from $s'$. If no such face is found along the root-to-leaf path in $T$, $s$ can be seen from $s'$, otherwise $s$ is blocked. The running time is easily seen to be $O(n \log^2 n)$, as preprocessing takes $O(n \log n)$ time and all update and query operations take at most $O(\log n)$ time per node visited. \hfill \(\square\)

Finally, we define the visibility neighborhood of $s$ as $V(s') := \bigcup_{π ∈ Π_{s'}} V(π,s')$.

**Corollary 1.** For each $s' \in S_{\text{sub}}$, we can compute $V(s')$ in $O(n \log^2 n)$ time.

### 2.2.2 Bounding the Degree of the Approximate Visibility Graph

Summarizing the above, we would like to connect each $s' \in S_{\text{sub}}$ with all $s \in V(s')$ by bridge edges. This approach can result in $|V(s')| \in Θ(n)$. The final challenge thus is to compute an approximation of $V(s')$ that results in sublinear-degree vertices in the visibility graph.
Definition 3 ([13]). Let $X \subset \mathbb{R}^3$ be a discrete point set and let $x$ be an arbitrary point in $X$. An approximate neighborhood $AH(x) := AH_\zeta(x)$ of $x$ w.r.t. $X$ is defined as a subset of $X \setminus \{x\}$, such that there exists a set of cones $\mathcal{C}(x) := \mathcal{C}_\zeta(x)$, with apex at $x$ and an angle $\zeta$ that covers $\mathbb{R}^3$, such that a point $x' \in X \setminus \{x\}$ belongs to the approximate neighborhood $AH(x) := AH_\zeta(x)$ iff there is a cone $C \in \mathcal{C}(x)$ such that $x'$ is the point in $C \cap X$ minimizing the distance from its orthogonal projection onto the axis of $C$ to $x$.

Lemma 7 ([13]). For $\zeta > 0$, approximate neighborhoods, each of size $\Theta(\zeta^{-2})$, for all points from $X$ can be computed in overall time $O(|X|/\zeta^2 \cdot \log^2(|X|))$

Stated in terms of Lemma 7, we compute the set $E_{\text{bri}}$ of bridge edges (see Section 2.2) in the weighted graph $G = (S_{\text{sub}}, E_{\text{loc}} \cup E_{\text{bri}})$ as follows: we iterate over all $s' \in S_{\text{sub}}$, compute $V(s')$ and then, for $\zeta := \sqrt{3} > 0$, approximate neighborhoods for the points in $V(s')$.

As a technicality, we wish to guarantee that bridge edges are not of local nature. Hence, we just consider edges that are longer than local edges (see Section 2.1.3). Thus, for $s' \in S_{\text{sub}}$, we define $A(s')$ as the $\sqrt{3}$-approximate neighborhood of $s'$ w.r.t. $V(s') \setminus B_{\frac{1}{\zeta^2} \cdot \sqrt{3} \cdot \text{afs}(s')}(s')$.

Corollary 2. For $s' \in S_{\text{sub}}$, we can compute $A(s')$ in $O(\max\{n \log^2 n, n/\delta \log^2 n\})$ time.

Algorithm 5 Compute the set $E_{\text{bri}}$ of bridge edges.

1: function ComputeBridgeEdges($S_{\text{sub}}, S, \delta, \text{afs}(\cdot)$)
2: $E_{\text{bri}} \leftarrow \emptyset$;
3: $S^\perp \leftarrow \text{ComputeCubeCenters}(S, \delta, \text{afs}(\cdot))$; \text{Use Algorithm 4}
4: for $s' \in S_{\text{sub}}$ do
5: \hspace{1em} $\Pi_{s'} \leftarrow \text{ComputePyramids}(s', S^\perp)$; \text{See Figure 1}
6: \hspace{1em} $V(s') \leftarrow \text{ComputeVisibleNeighborhood}(s', \Pi_{s'}, S_{\text{sub}}, S^\perp)$; \text{Use Corollary 1}
7: \hspace{1em} $A(s') \leftarrow \text{APXVisibleNeighborhood}(s', \delta, V(s'))$; \text{Use Lemma 7}
8: \hspace{1em} for $x \in A(s')$ do
9: \hspace{2em} $e \leftarrow (s', x)$; weight($e$) $\leftarrow |s'x|$; $E_{\text{bri}} \leftarrow E_{\text{bri}} \cup \{e\}$
10: return $E_{\text{bri}}$;

We show that $E_{\text{bri}}$ fulfills the requirements outlined at the beginning of Section 2.2.1:

Lemma 8. Let $S$ be an $\varepsilon$-sample and $E_{\text{bri}}$ the set of edges computed for a coarsened subsample $S'_{\text{sub}} \subseteq S$ according to Algorithm 2 and 5. Furthermore, let $\Sigma$ denote the solid bounded by $\Gamma$. Then, the following properties hold:

(BE1) For $s' \in S'_{\text{sub}}$, the visibility neighborhood $V(s')$ is a superset of the visibility edges of $S_{\text{sub}}$ w.r.t. $\Gamma$, i.e., $V(s')$ contains all edges $(s, s')$ such that $s \in S_{\text{sub}}$ and $ss' \cap \Sigma^o = \emptyset$ where $\Sigma^o$ denotes the interior of $\Sigma$.

For each $s \in S_{\text{sub}}$ with $|ss'| \geq \frac{1}{3} \cdot \sqrt{3} \cdot \text{afs}(s')$ there is an edge $(s, s'') \in E_{\text{bri}}$ such that $\angle(s, s', ss') \leq \delta$.\hfill\(\Box\)
(BE2) Let \((s, s') \in E_{\text{bri}}\) such that \(ss' \cap \Sigma^c \neq \emptyset\). The intersection is not too deep, hence, the shortcut taken not too short. More formally, for each edge \((s, s') \in E_{\text{bri}}\) and for any point \(x \in ss' \cap \Sigma\), there is a sample point \(s_x \in S_{\text{sub}}\) such that \(|xs_x| \leq 18 \cdot \delta \cdot \min\{afs(x), afs(s_x)\}\).

(IE3) Each sample point \(s' \in S_{\text{sub}}\) is incident to at most \(O(\sqrt{n})\) bridge edges.

Proof. (BE1): We prove (BE1) by showing that \(V(s')\) contains all sample points from \(S_{\text{sub}}\) that are visible from \(s'\) w.r.t. \(\Gamma\). In particular, we guarantee that each \(s'\)-skewed cube \(c := c_{s, x}\) lies inside the solid \(\Sigma\) bounded by \(\Gamma\). To do this, we first recall that \(c\) lies inside a ball \(B_{\delta \cdot afs(s)}(s^\dagger)\) with radius \(3.82 \cdot \delta \cdot afs(s)\) and centered in \(s^\dagger\), where \(s\) denotes the sample point corresponding to \(c\), see Lemma 5. Using a rather technical chain of reasoning (see Lemma 24), we can show \(s^\dagger \in \Sigma\) holds and that the distance between \(s^\dagger\) and \(\Gamma\) is lower-bounded by \(9 \cdot \delta \cdot afs(s)\). By the way Algorithm 4 constructs \(s^\dagger\) given \(s\), we know that \(|ss^\dagger| = 15 \cdot \delta \cdot afs(s)\) holds. As \(afs(\cdot)\) is 1-Lipschitz, it follows that \(afs(s^\dagger) \geq (1 - O(\delta)) \cdot afs(s)\) holds. This in turn implies \(|s^\dagger \Gamma| \geq 8 \cdot \delta \cdot afs(s)\).

Finally, the triangle inequality implies (BE1) as follows: Let \(z \in B_{\delta \cdot afs(s)}(s^\dagger)\) be chosen arbitrarily. We have \(|z\Gamma| \geq |s^\dagger \Gamma| - |s^\dagger z| \geq 8 \cdot \delta \cdot afs(s) - 3.82 \cdot \delta \cdot afs(s) > 0\).

(IE2): (To enhance the accessibility of this part of the paper, we defer the rather technical proof of this property to Section 3.1.2 (Lemma 22)).

(IE3): For each \(s' \in S_{\text{sub}}\), i.e., for each node of \(G\), we use the algorithm by Ruppert and Seidel [13], to compute an approximate \(\sqrt{\delta}\)-neighborhood for \(s'\) w.r.t. \(V(s')\) and connect \(s'\) to one representative per cone. With \(\zeta := \sqrt{\delta}\), the nodes of \(G\) thus are incident to \(O(\zeta^{-2}) = O(\delta^{-1})\) edges. Using again Lemma 2, we observe that \(\delta^{-1} \in O(\sqrt{n})\).

Combining Lemma 3 and Lemma 8, we see that the weighted graph \(G = (S_{\text{sub}}, E_{\text{loc}} \cup E_{\text{bri}})\) has a sublinear node degree and is well-suited to approximate the sought geodesic distances since the edges are either bridge edges, i.e., (approximate) visibility edges, or local edges, whose lengths are approximations of the geodesic distances of their endpoints.

2.3 Approximating All Distances / Runtime Analysis

We have described how to construct a weighted graph \(G := (S_{\text{sub}}, E_{\text{loc}} \cup E_{\text{bri}})\) on the coarsened set of sample points. While we can now use Dijkstra’s algorithm to compute shortest distances in this graph, i.e., between points in \(S_{\text{sub}}\), we also need to discuss how to compute distances between all sample points in \(S\) and not only between those in \(S_{\text{sub}}\).

The main idea is borrowed from the construction of spanner graphs based upon well-separated pair decompositions [10]. If a sample point \(s\) has been excluded from \(S_{\text{sub}}\) because it was found to lie inside a ball \(B_{3 \cdot \delta \cdot afs(s)}(s')\) of some sample point \(s' \in S_{\text{sub}}\), the distances to/from \(s'\) are good enough approximations of the distances to/from \(s\) as long as the destination is “far away”; otherwise, we use the Euclidean distance as an approximation.
Algorithm 6 Deriving an approximation $L(\cdot, \cdot)$ of $L^\star_\Gamma(\cdot, \cdot)$ from $G$.

1: function APXDistancesFromGraph($S$, $G$)
2: Compute shortest path distances $L_G(s_1, s_2)$ for all $s_1, s_2 \in S^{\text{sub}}$.
3: for all $s \in S$ do
4: $\nu_s \leftarrow$ sample point in $S^{\text{sub}}$ closest to $s$; \hspace{1em} \triangleright \hspace{0.5em} \nu_s = s$ for all $s \in S^{\text{sub}}$
5: for all $s_1, s_2 \in S$ do
6: if $|s_1 s_2| \leq \frac{1}{\delta} \cdot \min\{afs(s_1), afs(s_2)\}$ then
7: $L(s_1, s_2) \leftarrow |s_1 s_2|$; \hspace{1em} \triangleright \hspace{0.5em} \text{Approximate by Euclidean distance}
8: else
9: $L(s_1, s_2) \leftarrow L_G(\nu_{s_1}, \nu_{s_2});$ \hspace{1em} \triangleright \hspace{0.5em} \text{Approximate using closest points in } S^{\text{sub}}
10: return $L(\cdot, \cdot);$ 

We now have collected all ingredients needed to analyze the running time of Algorithm 1.

Lemma 9. Algorithm 1 has a running time of $O(n^{5/2} \log^2 n)$.

Proof. To recall Algorithm 1: we first compute a control function (Lemma 1) and approximate the local feature size (Observation 1). Based upon these results, we can (asymptotically) lower-bound the local feature size. Since we need the poles for this, we construct the Voronoi diagram of all points, which can be done in $O(n^2)$ time. In the next phase of the algorithm, we work with a coarsened subsample $S^{\text{sub}}$ which can be constructed in $O(n^2)$ time as well (Algorithm 2). Since we compute the set of local edges by iterating over all pairs of points in $S^{\text{sub}}$ (Algorithm 3), this step takes $O(n^2)$ time as well. Computing the set of bridge edges takes $O(n \cdot \max\{n \log^2 n, n/\delta \log^2 n\}) \leq O(n^{5/2} \log^2 n)$ time (Corollary 2, Lemma 2). The running time of the final step (Algorithm 6) is dominated by the $\Theta(n)$-fold invocation of Dijkstra’s algorithm on a graph with $O(n)$ vertices and $O(n^{3/2})$ edges (Lemma 3 (LE3), Lemma 8 (BE3)). Using an efficient priority queue implementation, the running time for this step is $O(n^{5/2} \log n)$. Hence, the overall running time of Algorithm 1 is $O(n^{5/2} \log^2 n)$.

3 Analysis of the Approximation Quality

For ease of presentation, we will refer to $L(s_1, s_2)$ as the edge length between $s_1$ and $s_2$.

Lemma 10. $L(\cdot, \cdot)$ is a $(1 \pm O(\sqrt{\varepsilon}))$-approximation of $L^\star_\Gamma(\cdot, \cdot)$.

To prove Lemma 10, we first relate the value of $\delta$ to $\varepsilon$ (Lemma 11). In Subsection 3.1, we then show $L(\cdot, \cdot) \geq (1 - O(\sqrt{\varepsilon})) \cdot L^\star_\Gamma(\cdot, \cdot)$. Finally, we show $L(\cdot, \cdot) \leq (1 + O(\sqrt{\varepsilon})) \cdot L^\star_\Gamma(\cdot, \cdot)$ (see Subsection 3.2). This concludes the proof of Lemma 10 and, hence, of Theorem 1.

Lemma 11. $\delta \in O(\varepsilon)$.
Proof. We combine Lemma 1 and Aichholzer et al.’s observation that $lfs(s) \leq 1.2802 \cdot afs(s)$ holds for all $s \in S$ [1, Lemma 5.1] and upper bound $\delta$ as follows: $\delta = \max_{s \in S} \frac{v(s)}{afs(s)} \leq \max_{s \in S} \frac{1.19\varepsilon \cdot lfs(s)}{1.2802 - 1} \cdot lfs(s) \leq O(\varepsilon)$. \hfill $\square$

3.1 Lower-Bounding the Edge Length

To ensure $L((s_1, s_2)) \geq (1 - O(\varepsilon))L^*_{\Gamma}(s_1, s_2)$ for all $s_1, s_2 \in S$, we consider a shortest path $\phi$ in the distance graph $G$ between $v_{s_1}$ and $v_{s_2}$. We then construct a curve $\gamma$ between $s_1$ and $s_2$ in the free space $\Lambda := \mathbb{R}^3 \setminus \Sigma$ such that $|\gamma| \leq (1 + O(\sqrt{\varepsilon})) \cdot |\phi|$, or, equivalently, $(1 - O(\sqrt{\varepsilon})) \cdot |\gamma| \leq |\phi|$ holds. To show this, we separately consider the local edges and the bridges edges on $\phi$.

3.1.1 Lower-Bounding the Edge Length of Local Edges (LE1)

Lemma 16 gives the lower bound for the length of local edges. In a previous paper [14], we proved a corresponding lower bound for edges whose lengths are related to $\varepsilon$ and $lfs(\cdot)$:

**Lemma 12 ([14, Lemma 20]).** For $x, y \in \Gamma$ with $|xy| \leq \sqrt{\varepsilon} \cdot \min\{lfs(x), lfs(y)\}$, we have $L_{\Gamma}(x, y) \leq (1 + O(\varepsilon)) \cdot |xy|$, where $L_{\Gamma}(x, y)$ is the geodesic distance of $x$ and $y$ on $\Gamma$.

Lemma 12 cannot be applied directly to a local edge $(p, q) \in E_{loc}$, since $|pq|$ depends on $\delta$ and $afs(\cdot)$ instead of $\varepsilon$ and $lfs(\cdot)$. To extend this result to local edges, we can show that a similar statement also applies to free-space geodesic distances in our case:

**Lemma 13.** For $x_1, x_2 \in \Lambda$ with $x_1 \neq x_2$ and $|x_1 x_2| \leq \sqrt{\varepsilon} \cdot \min\{lfs(x_1), lfs(x_2)\}$, we have $L^*_{\Gamma}(x_1, x_2) \geq (1 + O(\varepsilon)) \cdot |x_1 x_2|$.

**Proof.** Recall that $\Lambda := \mathbb{R}^3 \setminus \Sigma$, i.e., $x_1$ and $x_2$ cannot lie in the interior of the solid bounded by $\Gamma$. If $x_1$ and $x_2$ both lie on $\Gamma$, Lemma 12 directly implies $L^*_{\Gamma}(x_1, x_2) \leq L_{\Gamma}(x_1, x_2)$ if $x_1, x_2 \in \Gamma$. We thus assume w.l.o.g. that $x_2$ does not lie on $\Gamma$.

Let $y_1 \in x_1 x_2$ be the point such that $x_1 y_1 \setminus \{x_1, y_1\} \cap \Sigma = \emptyset$, i.e., the interior of $x_1 y_1$ is the longest part of $x_1 x_2$ “starting” at $x_1$ and not intersecting $\Gamma$ (see Figure 2). If $y_1 = x_2$, we know in particular that $y_1 \neq x_1$, hence $x_1 x_2$ is fully contained in $\Lambda$ and we have $L^*_{\Gamma}(x_1, x_2) = |x_1 x_2| \leq (1 + O(\varepsilon)) \cdot |x_1 x_2|$ as claimed.

To complete the proof, we this consider the case that $y_1 \neq x_2$ holds. We define $\zeta := |x_1 y_1| / lfs(x_1)$. By this definition, $|x_1 y_1| = \zeta \cdot lfs(x_1)$ holds, which implies $\zeta \leq \sqrt{\varepsilon}$. From the upper bound on $|x_1 x_2|$, we obtain $|y_1 x_2| \leq \sqrt{\varepsilon} \cdot lfs(x_1) - \zeta \cdot lfs(x_1) = (\sqrt{\varepsilon} - \zeta) \cdot lfs(x_1)$

Figure 2: Construction of a curve between $x_1$ and $x_2$ in the configuration of Lemma 13.
and \( \text{afs}(y) \geq \text{afs}(x) - \zeta \cdot \text{afs}(x) = (1 - \zeta) \text{afs}(x) \). So we get \( |y_1x_2| \leq \frac{\sqrt{\varepsilon} \cdot \zeta}{1-\zeta} \cdot \text{afs}(y) \). As \( \sqrt{\varepsilon} \leq \varepsilon < 1 \), we get \( 1 - \frac{\zeta}{\sqrt{\varepsilon}} \leq 1 - \zeta \) or, equivalently, \( \frac{1-\zeta/\sqrt{\varepsilon}}{1-\zeta} \leq 1 \). This in turn implies \( \frac{\sqrt{\varepsilon} \cdot \zeta}{1-\zeta} \cdot \text{afs}(y) \leq \sqrt{\varepsilon} \cdot \text{afs}(y) \).

Still assuming w.l.o.g. that \( x_2 \) does not lie on \( \Gamma \), we define \( y_2 \in x_1x_2 \) to be the point such that \( y_2x_2 \setminus \{y_2, x_2\} \cap \Sigma = \emptyset \). By construction, \( |y_1y_2| \leq |y_1x_2| \) and thus \( |y_1y_2| \leq \sqrt{\varepsilon} \cdot \text{afs}(y_1) \). Lemma 12 then implies \( L_r(y_1, y_2) \leq \left(1 - O(\varepsilon)^{-1}\right) \cdot |y_1y_2| \). Combining a curve \( \gamma_{y_1y_2} \) realizing this distance with \( x_1y_1 \) and \( y_2x_2 \) yields a curve connecting \( x_1 \) and \( x_2 \) in \( \Lambda \) whose length is no larger than \( (1 - O(\varepsilon)^{-1}) \cdot |x_1x_2| \).

\[ \square \]

Lemma 13 allows us to iteratively construct the curve \( \gamma \) discussed above by connecting points on \( \Gamma \) and in \( \Lambda \). To bound the length of this curve, we will resort to the following technical lemma:

**Lemma 14** ([5]). For \( s \in S \), let \( n_s \) be the normal of \( \Gamma \) in \( s \) and \( p \) a pole of \( s \). Then we have \( \angle(sp, n_s) \leq \arcsin\left(\frac{\varepsilon}{1-\varepsilon}\right) \).

We use \( \gamma \) to prove Lemma 16 and, hence, to complete the proof of Lemma 3 (LE1). As a technical tool needed in the proofs of Lemma 16 and the auxiliary Lemma 15, we need the following corollary of a lemma by Aichholzer et al. [1, Lemma 4.2]:

**Corollary 3.** Let \( s \in S \) and \( p^* \) a pole point of \( s \). Then we have \( |p^*y| \geq |p^*s| - \varepsilon \cdot \text{afs}(y) \) and \( |p^*y| \geq \left(\sqrt{1 - 4\left(\varepsilon^2 - \frac{\varepsilon^4}{4}\right)} - \varepsilon^2\right) |sp^*| \geq (1 - 3\varepsilon)|sp^*| \) for any \( y \in \Gamma \).

For \( p_i, i \in \{1, 2\} \), let \( p_i^+ \) be the inner pole point and \( p_i^- \) be the outer pole point of \( p_i \). The following lemma states that the normal vectors of two points in \( S \) “close” to each other are “not too different”.

**Lemma 15.** For two sample points \( s_1, s_2 \in S \) with \( |s_1s_2| \leq O(\sqrt{\varepsilon}) \cdot \min\{\text{afs}(s_1), \text{afs}(s_2)\} \), we have \( \angle(s_1p_1^+, s_2p_2^+) \leq O(\sqrt{\varepsilon}) \).

**Proof.** We start by considering the angle between \( s_1s_2 \) and \( s_1p_1^+ \), i.e., the angle in the plane induced by these three points. Using a chain of reasoning similar to the one we will apply in the proof of Lemma 16, we obtain that in this plane \( |\angle(s_1, s_2, p_1^+) - 90^\circ| \leq O(\sqrt{\varepsilon}) \) holds. Analogously, we obtain that in the plane induced by \( s_1, s_2, \) and \( p_2^+ \), \( |\angle(s_1, s_2, p_2^+) - 90^\circ| \leq O(\sqrt{\varepsilon}) \) holds.

We now need to show that the angle between these two planes considered is not too large, i.e., that the torsion between \( s_1p_1^+ \) and \( s_2p_2^+ \) is small. For this, we construct balls centered at the inner and outer pole points such that these balls do not contain any part of \( \Gamma \). Using Lemma 14, we show that \( s_i p_i^+ \) and \( s_i p_i^- \), \( i = 1, 2 \), are almost collinear. We then show that two balls centered at poles of different points cannot intersect. From this, we construct a worst-case configuration to upper-bound the torsion.
More formally, we define $B_1^+ := B_{(1-3\varepsilon)|s_1 p_1^+|}(p_1^+)$, $B_1^- := B_{(1-3\varepsilon)|s_1 p_1^-|}(p_1^-)$, $B_2^+ := B_{(1-3\varepsilon)|s_2 p_2^+|}(p_2^+)$, and $B_2^- := B_{(1-3\varepsilon)|s_2 p_2^-|}(p_2^-)$ be the shrunken polar balls under consideration. Corollary 3 implies none of these balls has a point with $\Gamma$ in common. Furthermore, we know that $B_1^+$ and $B_2^-$ lie on different sides of $\Gamma$ which implies that $B_1^+$ and $B_2^-$ are not allowed to intersect each other.

To construct a worst-case situation, we assume that the angles $|\angle(s_1, s_2, p_1^+) - 90^\circ|$ and $|\angle(s_2, s_1, p_2^+) - 90^\circ|$ are maximized, i.e., that they lie in $\Theta(\sqrt{\varepsilon})$. As observed by Aichholzer et al. [1] in the proof of their Lemma 4.2, a large torsion of $s_1 p_1^+$ and $s_2 p_2^+$ corresponds to an intersection of one outer polar ball and one inner polar ball. However, by the choice of the radii of the balls under consideration, Lemma 3 guarantees that the balls $B_1^+$ and $B_2^-$ at most touch each other.

To bound the torsion, we consider the plane $E$ containing $s_1$ and $s_2$ such that $\angle(s_1 p_1^+, E) = \angle(E, s_2 p_2^-)$ and $p_1^+$ and $p_2^-$ lie on the same side of $E$ (such a plane exists by a continuity argument regarding the increase and decrease of these angles while rotating a plane around $s_1 s_2$).

As discussed above, we can assume that the balls $B_1^+$ and $B_2^-$ are touching each other. Thus, decreasing the radii of $B_1^+$ and $B_2^-$ while at the same time maintaining contact between the shrinking balls increases the angle $\angle(s_1 p_1^+, s_2 p_2^-)$. Consequently, because we want to upper-bound the angle $\angle(s_1 p_1^+, s_2 p_2^-)$, we can assume w.l.o.g. that $|s_1 p_1^+| = |s_2 p_2^-| = \min\{|s_1 p_1^+|, |s_2 p_2^-|\}$ holds. This implies that the distance between $p_1^+$ and the plane $E$ is equal to the distance between $p_2^-$ and $E$ because $\angle(s_1 p_1^+, E_2) = \angle(s_2 p_2^-, E_2)$. As $p_1^+$ and $p_2^-$ lie on the same side of $E$, we can now consider the plane $E_2$ that lies parallel to $E$ such that $p_1^+, p_2^- \in E_2$. Furthermore, we consider the circles $C_1^+ := \partial B_1^+ \cap E_2$ and $C_2^- := \partial B_2^- \cap E_2$. As we assumed for a worst-case configuration that $B_1^+$ and $B_2^-$ are touching each other, we obtain that $C_1^+$ and $C_2^-$ are touching each other in the plane $E_2$, see Figure 3.

By the assumption of the lemma, we have $|s_1 s_2| \leq O(\sqrt{\varepsilon}) \cdot \min\{|afs(s_1), afs(s_2)|\}$. Thus, we can show $|s_1 s_2| \leq O(\sqrt{\varepsilon}) \cdot \min\{|afs(s_1), afs(s_2)|\} \leq O(\sqrt{\varepsilon}) \cdot \min\{|s_1 p_1^+, |s_2 p_2^-|\}$. As $C_1^+$ and $C_2^-$ are touching each other, we have $|p_1^+ p_2^-| = 2 \cdot (1 - 3 \varepsilon) \cdot \min\{|s_1 p_1^+, |s_2 p_2^-|\}$,

Figure 3: Upper bound construction for the torsion of $s_1 p_1^+$ and $s_2^+ p_2^+$ in the proof of Lemma 15.
see Figure 3.

To continue the proof, we let $q_1$ denote the orthogonal projections of $s_1$ onto $E_2$ and let $q_2$ denote the orthogonal projection of $s_2$ onto $E_2$. The segment between the two points $s_1$ and $s_2$ is parallel to $E_2$ because the segment between $s_1$ and $s_2$ lies parallel to $E$ which in turn lies parallel to $E$. Hence, we have $|q_1q_2| = |s_1s_2|$. The point symmetry of the configuration of the points $p_1^+$, $q_1$, $q_2$, and $p_2^-$ w.r.t. the midpoint $m$ of the segment between $q_1$ and $q_2$ implies that the circles $C_1^+$ and $C_2^-$ are touching each other in $m$. Let $c_1$ be the orthogonal projection of $p_1^+$ onto the line $\ell$ that is induced by $q_1$ and $q_2$. As the angle $\angle(p_1^+, s_1, s_2)$ is upper-bounded by $O(\sqrt{\varepsilon}) + 90^\circ$, we obtain $|q_1c_1| \leq \arcsin(O(\varepsilon)) \cdot |s_1p_1^+| \leq O(\varepsilon) \cdot |s_1p_1^+| = O(\varepsilon) \cdot \min\{|s_1p_1^+|, |s_2p_2^-|\}$ (as usual, we exploit that $\arcsin(O(\varepsilon)) \leq O(\varepsilon)$ for $\varepsilon$ sufficiently small). Furthermore, by assumption of the lemma, we have $|q_1q_2| = |s_1s_2| \leq O(\sqrt{\varepsilon}) \cdot \min\{|afs(s_1), afs(s_2)| \leq O(\sqrt{\varepsilon}) \cdot \min\{|s_1p_1^+|, |s_2p_2^-|\}$ which implies $|q_1m| \leq O(\sqrt{\varepsilon}) \cdot \min\{|s_1p_1^+|, |s_2p_2^-|\}$ because $|q_1m| = \frac{1}{\varepsilon} \cdot |q_1q_2|$. Thus, we have $\angle(c_1, p_1^+, m) \leq \arcsin\left(\frac{|c_1q_1| + |q_1m|}{|p_1^+m|}\right) \leq \arcsin(O(\sqrt{\varepsilon})) \leq O(\sqrt{\varepsilon})$. This yields $\frac{|c_1q_1|}{|p_1^+m|} \geq \cos(O(\sqrt{\varepsilon}))$ which in turn implies $|p_1^+c_1| \geq (1 - O(\varepsilon)) \cdot |p_1^+m|$. Let $d_1$ be the point from the line induced by the two sample points $s_1$ and $s_2$ and whose orthogonal projection onto $\ell$ is equal to $c_1$. As $c_1$ is the orthogonal projection of $d_1$ onto $\ell$, we obtain that $c_1$ is also the orthogonal projection of $d_1$ onto $E_2$. Thus the angle $\angle(d_1, c_1, p_1^+)$ is equal to $90^\circ$. Thus, the angle $\angle(c_1, p_1^+, d_1)$ is upper-bounded by $\arccos\left(\frac{|c_1d_1|}{|p_1^+d_1|}\right) \leq \arccos(1 - O(\varepsilon)) \leq O(\sqrt{\varepsilon})$. As the angle $\angle(c_1, p_1^+, d_1)$ is equal to the angle $\angle(E_2, p_1^+, s_1)$, we obtain $\angle(E_2, p_1^+, s_1) \leq O(\sqrt{\varepsilon})$ as well.

A symmetric argument implies $\angle(E_2, p_2^+, s_2) \leq O(\sqrt{\varepsilon})$ which in turn implies that the torsion between $s_1p_1^+$ and $s_2p_2^+$ is upper-bounded by $O(\sqrt{\varepsilon})$, because Lemma 14 implies $\angle(p_2^+, s_2, p_2^-) - 180^\circ \leq O(\varepsilon)$.

Finally, we upper-bound the angle between the segments $s_1p_1^+$ and $s_2p_2^+$: Let $E_1$ and $E_2$ be the planes orthogonal to $\ell$ such that $s_1 \in E_1$ and $s_2 \in E_2$. We already observed $\angle(p_1^+, s_1, s_2) + 90^\circ \leq O(\sqrt{\varepsilon})$ and $\angle(p_2^+, s_2, s_1) + 90^\circ \leq O(\sqrt{\varepsilon})$ which implies $\angle(s_1p_1^+, E_1), \angle(s_2p_2^+, E_2) \leq O(\sqrt{\varepsilon})$. Hence, we can bound $\angle(s_1p_1^+, s_2p_2^+) \leq \angle(s_1p_1^+, E_2) + \angle(s_1p_1^+, E_1) + \angle(s_2p_2^+, E_2) + \angle(s_2p_2^+, E_1) \leq O(\sqrt{\varepsilon})$ which concludes the proof.

If, when measured in terms of the approximations of the sampling density and of the local feature size, two sample points lie “close to each other”, their geodesic distances is “not too large”. More formally, we have the following lemma:

**Lemma 16.** For $s_1, s_2 \in S^\text{sub}$ with $|s_1s_2| \leq \frac{1}{2} \cdot \sqrt{\varepsilon} \cdot \min\{|afs(s_1), afs(s_2)|$, we have $L_1^r(s_1, s_2) \leq (1 + O(\sqrt{\varepsilon})) \cdot |s_1s_2|$, i.e., $L(s_1, s_2) \geq (1 - O(\sqrt{\varepsilon})) \cdot L_1^r(s_1, s_2)$.

**Proof.** If $|s_1s_2| \leq \frac{1}{2} \cdot \sqrt{\varepsilon} \cdot \min\{|fs(s_1), fs(s_2)|$, i.e., if the points lie close to each other even with respect to the non-approximated sampling density and local feature size, the proof immediately follows from Lemma 13. Thus, we assume $|s_1s_2| > \frac{1}{2} \cdot \sqrt{\varepsilon} \cdot \min\{|fs(s_1), fs(s_2)|$(\). The high-level approach of the proof is as follows: Let $p_1^+, p_2^+, p_1^-, p_2^-$ be the outer and inner poles corresponding to $s_1$ and $s_2$, see Figure 4. We will construct a curve
\(\gamma_{s_1, s_2} \subset \left( B_{|s_1 p_1^+|}(p_1^+) \cup B_{|s_2 p_2^+|}(p_2^+) \right) \cap \Lambda\) connecting \(s_1\) and \(s_2\) in the free space \(\Lambda\) and then show that \(|\gamma_{s_1, s_2}| \leq (1 + O(\sqrt{\varepsilon})) \cdot |s_1 s_2|\) holds; this will conclude the proof.

For the construction of the curve \(\gamma_{s_1, s_2}\), we proceed as follows: Let \(E\) be the plane such that \(s_1, s_2, p_2^+ \in E\) and consider the intersection of the polar balls centered at \(p_1^+\) and \(p_2^+\). We define \(C_1^+ := \partial B_{|s_1 p_1^+|}(p_1^+) \cap E\) and \(C_2^+ := \partial B_{|s_2 p_2^+|}(p_2^+) \cap E\). Let \(c_1^+\) and \(c_2^+ = p_2^+\) be the centers of \(C_1^+\) and \(C_2^+\). By definition, we have \(c_2^+ \in \Lambda\). We show that \(c_1^+\) lies in \(\Lambda\).

To show that \(c_1^+ \in \Lambda\), we proceed as follows: Since \(s_1, p_1^+, s_2 \in E\), we have \(\angle(E, s_1 p_1^+) = 0\). Furthermore, Lemma 14 implies \(\angle(p_1^+, s_1, p_1^-) \leq O(\varepsilon)\) and thus we obtain \(|\pi - \angle(E, s_1 p_1^-)| \leq O(\varepsilon)|\) (as usual, we assume that \(\varepsilon\) is sufficiently small such that \(\arcsin(\varepsilon) \leq O(\varepsilon)\)). Since, by assumption, \(|s_1 s_2| \leq \frac{1}{4} \cdot \sqrt{\varepsilon} \cdot \min\{afs(s_1), afs(s_2)\}\) and since \(\sqrt{\varepsilon} \in O(\sqrt{\varepsilon})\) (Lemma 11), Lemma 15 implies \(\angle(E, s_1 p_1^+), \angle(E, s_1 p_1^-) \leq O(\sqrt{\varepsilon})\). This implies that \(\angle(s_1 c_1^+, s_1 p_1^+) = \angle(E, s_1 p_1^+)\) can be assumed to be upper-bounded by a constant, say, \(\pi/6\). Hence, \(|p_1^+ c_1^+| \leq \sin(\pi/6) \cdot |s_1 p_1^+| = \frac{1}{2}|s_1 p_1^+|\). Furthermore, Corollary 3 implies that \(|y p_1^+| \geq |s_1 p_1^+| - \varepsilon \cdot \text{afs}(y)\) for any \(y \in \Gamma\). For the sake of contradiction, we assume that \(c_1^+\) does not lie inside the free space \(\Lambda\). As we know (by definition of an outer pole) that \(p_1^+ \in \Lambda\) holds, we then know that there exists some point \(x \in \Gamma \cap c_1^+ p_1^+\). For this point, we would have \(|p_1^+ x| \leq |p_1^+ c_1^+| \leq \frac{1}{2}|s_1 p_1^+|\). Aichholzer et al. [1, Lemma 5.1] observed that for all \(s \in S\) the inequality \(\text{afs}(s) \leq 1.2802 \cdot \text{afs}(s)\) holds. As \(S\) is an \(\varepsilon\)-sample, we would find a sample point \(s_x \in \varepsilon\)-sample such that \(|x s_x| \leq \varepsilon \cdot \text{ifs}(x) \leq 1.2802 \cdot \varepsilon \cdot \text{afs}(x)\). As \(\text{ifs}(x) \leq |c_1^+ p_1^+| \leq \frac{1}{2}|s_1 p_1^+|\), the triangle inequality shows that \(|p_1^+ s_x| < |p_1^+ s_1|\), i.e., that the sample point \(s_x\) lies inside the polar ball centered at \(p_1^+\). As, by definition, no sample point can lie inside any polar ball, we have a contradiction. Hence, we have shown that \(c_1^+ \in \Lambda\).

Lemma 15 implies \(\angle(s_1 p_1^+, s_2 p_2^+) \leq O(\sqrt{\varepsilon})\) for \(s \in \{+, -\}\). Using an argument very similar to the one used for upper-bounding \(|p_1^+ c_1^+|\), we can lower-bound the distances \(|s_1 c_1^+|\) and \(|s_2 c_2^+|\) and thus lower-bound the radii of \(C_1^+\) and \(C_2^+\) by \(\frac{1}{2} \cdot \min\{afs(s_1), afs(s_2)\}\).

Because the pole points \(p_1^+, p_1^-, p_2^+\), and \(p_2^-\) are defined as Voronoi vertices, we have \(s_1, s_2 \notin C_1^+ \cup C_1^- \cup C_2^+ \cup C_2^-\). Hence, we can show \(|\angle(s_1, s_2, c_2^+) - 90^\circ| \in O(\varepsilon)\) and \(|\angle(s_1, s_2, c_1^+) - 90^\circ| \leq O(\varepsilon)\). In particular, Lemma 14 implies \(\angle(p_1^+, s_1, p_1^-), \angle(p_2^+, s_2, p_2^-) \geq \varepsilon\).
180° − \frac{\varepsilon}{1 - \varepsilon}. Furthermore, the angle \( \angle(s_2, s_1, p_1^+) \) is maximized when \( p_1^- \) is rotated around \( s_1 \) into the direction of \( s_2 \) such that \( s_2 \in \partial B_{|p_1^-|}(p_1^-) \) hold and \( p_1^+ \) is rotated around \( s_1 \) and away from \( s_2 \) such that \( \angle(p_1^+, s_1, p_1^-) = 180° - \frac{\varepsilon}{1 - \varepsilon} \) is obtained. In this configuration, we can use elementary trigonometric calculations to show \( \angle(s_1, s_2, p_2^+) = 90° \leq O(\sqrt{\varepsilon}) \). From \( \angle(s_1, s_2, c_2') = 90° \leq \angle(s_1, s_2, p_2^+) = 90° \), we then obtain \( \angle(s_1, s_2, c_2') - 90° \leq O(\sqrt{\varepsilon}) \). A symmetric argument implies \( \angle(s_2, s_1, c_1^+) - 90° \leq O(\sqrt{\varepsilon}) \).

The bounds \( \angle(s_1, s_2, p_2^+) - 90° \leq O(\varepsilon) \) and \( \angle(s_2, s_1, p_1^+) - 90° \leq O(\varepsilon) \) derived above imply that \( C_1^+ \) and \( C_2^+ \) intersect in two points; let \( z \) be the one closer to \( s_1, s_2 \), see Figure 4. Since the radii of \( C_1^+ \) and \( C_2^+ \) are lower-bounded by \( \frac{1}{2} \cdot \min\{afs(s_1), afs(s_2)\} \), since \( \angle(s_1s_2, s_1p_1^+) = 90° \leq O(\varepsilon) \), and since \( \angle(s_1s_2, s_1p_1^+) - 90° \leq O(\varepsilon) \), we obtain \( \angle(z, s_1, s_2) \leq \varepsilon \).

We now consider the curve \( \gamma_z \) composed of the two subcurves on \( C_1^+ \) and \( C_2^+ \) between \( s_1, s_2 \) and \( z \) and between \( z \) and \( s_2 \), see Figure 4. As \( \angle(z, s_1, s_2) \leq (1 + O(\sqrt{\varepsilon})) \cdot |\gamma_z| \). This will result in \( |\gamma_{s_1s_2}| \leq (1 + O(\sqrt{\varepsilon})) \cdot |\gamma_z| \).

For each \( x \in \gamma_z \), there is an \( x' \in \Lambda \) such that \( |xx'| \leq \frac{3\varepsilon}{1 - 3\varepsilon} \cdot \min\{|afs(x), afs(x')|\} \). Corollary 3 implies that \( |yp_{1+}^x| \geq |p_{1+}^x s_1| - \varepsilon \cdot |afs(y)| \) and \( |yp_{2+}^x| \geq |p_{2+}^x s_2| - \varepsilon \cdot |afs(y)| \) hold for all \( y \in \Gamma \). W.l.o.g., we assume \( x \in C_1^+ \). For the sake of contradiction assume \( |xx'| > 3 \cdot \varepsilon \cdot |afs(x)| \). This implies \( x \notin \Lambda \). Since \( c_1^+ \in \Lambda \) and since \( \Gamma \) separates the interior of \( \Lambda \) from the solid \( \Sigma \) bounded by \( \Gamma \), there is an \( x' \in \Gamma \cap xc_1^+ \). By assumption, we obtain \( |xx'| > 3 \cdot \varepsilon \cdot |afs(x)| \) which implies \( |p_{1+}^x| < |s_1p_{1+}^x| - \frac{3}{2} \cdot \varepsilon \cdot |afs(x)| \), because \( \angle(p_{1+}^x, s_1, c_1^+) = \angle(p_{1+}^x, s_1, c_1^+) \leq \pi/6 \) (the point \( c_1^+ \) is the orthogonal projection of \( p_1^+ \) onto \( E \), hence, \( |c_1^+ s_1| = |c_1^+ x| \)). As \( afs(\cdot) \) is 1-Lipschitz, we obtain \( |p_{1+}^x| < |s_1p_{1+}^x| - \varepsilon \cdot |afs(x')| \) which is a contradiction to Corollary 3. Thus we obtain \( |xx'| \leq 3 \cdot \varepsilon \cdot |afs(x)| \) which, using the 1-Lipschitz property of \( afs(\cdot) \), can be upper-bounded by \( \frac{3\varepsilon}{1 - 3\varepsilon} \cdot \min\{|afs(x), afs(x')|\} \). A symmetric argument shows that there is an \( x' \in \Lambda \) such that \( |xx'| \leq \frac{3\varepsilon}{1 - 3\varepsilon} \cdot \min\{|afs(x), afs(x')|\} \) holds for all \( x \in \gamma_z \). For two points \( x, y \in \gamma_z \), let \( \phi_{xy} \) be the subcurve of \( \gamma_z \) between \( x \) and \( y \) (see Figure 4). Let \( x \in \gamma_z \) such that \( |xx'| \leq \frac{3\varepsilon}{1 - 3\varepsilon} \cdot \min\{|afs(x), afs(x')|\} \). Let \( x' \in \Lambda \) such that \( |xx'| \leq \frac{3\varepsilon}{1 - 3\varepsilon} \cdot \min\{|afs(x), afs(x')|\} \). The point \( x \) exists because we assumed \( |s_1s_2| > \frac{1}{2} \cdot \sqrt{\varepsilon} \cdot \min\{|afs(s_1), afs(s_2)|\} \) above, see (1). This implies \( |s_1x'| \leq \sqrt{\varepsilon} \cdot \min\{|afs(s_1), afs(x')|\} \) and thus \( |xx'| \leq O(\sqrt{\varepsilon}) \cdot |\phi_{s_1x}| \). Thus, Lemma 13 implies \( l_{+}^*(s_1, x') \leq (1 + O(\sqrt{\varepsilon})) \cdot |\phi_{s_1x}| \).

We repeat this construction for \( s_1 \leftarrow x \) until \( x = s_2 \). Doing so, we obtain a curve \( \gamma_{s_1s_2} \) composed of the subcurves constructed as discussed above such that \( |\gamma_{s_1s_2}| \leq (1 + \sqrt{\varepsilon})|\gamma_z| \); see Figure 4.

\[ \square \]

### 3.1.2 Lower-Bounding the Edge Length of Bridge Edges

Lemma 17 gives the lower bound for the length of bridge edges.

**Lemma 17.** For \((s, q) \in E_{bri} \subseteq S_{sub} \times S_{sub} \), we have \( L_{+}^*(s, q) \leq (1 + O(\sqrt{\varepsilon})) \cdot |sq| \).

To be able to prove Lemma 17, we need to ensure that, given some \((s, q) \in E_{bri}, for
each \( x \in \overline{s\ell q} \cap \Sigma \) there is a \( s_x \in \mathcal{S}^\text{sub} \) such that \( |xs_x| \leq O(\varepsilon) \cdot \min\{\text{afs}(s), \text{afs}(q)\} \). Applying Lemma 16 multiple times then yields \( L_\alpha^*(s, q) \leq (1 + O(\sqrt{\varepsilon})) \cdot |sq| \). To ensure the existence of such an \( s_x \) with the properties mentioned above, we need to prove that \( x \) lies not “too deep” inside \( \Sigma \), see Lemma 8 (BE2). For this, we will consider the restricted Delaunay triangulation of \( S \) w.r.t. \( \Gamma \).

**Definition 4 ([5]).** Let \( t \) be the triangle induced by three sample points \( s_1, s_2, s_3 \in S \). \( t \) is an element of the restricted Delaunay triangulation \( T \) iff \( \text{Vor}_S(s_1) \cap \text{Vor}_S(s_2) \cap \text{Vor}_S(s_3) \cap \Gamma \neq \emptyset \).

To show that the bridge edges do not penetrate \( \Sigma \) “too deeply” we first push each sample point into the direction of its inner pole point and the triangulate these pushed pole points corresponding to the Delaunay triangulation \( T \) of \( S \) restricted to \( \Gamma \). In other words: \( T^\downarrow \) is the triangulation induced from \( T \) by mapping \( S \) to \( S^\downarrow \), where \( S^\downarrow \) is computed by Algorithm 4.

**Definition 5.** Let \( T \) be the Delaunay triangulation of \( S \) restricted to \( \Gamma \). Furthermore, let \( \Delta(s_1, s_2, s_3) \) be the triangle with the corners \( s_1, s_2, s_3 \). For \( i = 1, 2, 3 \), let \( s_i^\downarrow \) be the point resulting from pushing \( s_i \) into the direction of its inner pole point \( p_i^\downarrow \) about a distance of \( 15 \cdot \delta \cdot \text{afs}(s_i) \cdot \frac{p_i^\downarrow}{|p_i^\downarrow|} \), see Algorithm 4. We define \( T^\downarrow := \{\Delta(s_1^\downarrow, s_2^\downarrow, s_3^\downarrow) \mid \Delta(s_1, s_2, s_3) \in T\} \) where \( \Delta(s_1^\downarrow, s_2^\downarrow, s_3^\downarrow) \) is the triangle with the corners \( s_1^\downarrow, s_2^\downarrow, s_3^\downarrow \).

In order to show that bridge edges do not shortcut too much, we need to show that bridge edges do not penetrate too deeply the solid \( \Sigma \) bounded by \( \Gamma \). In order to do this, we show that \( T^\downarrow \) is covered by the union of the \( \ell' \)-skewed cubes. Assuming that we can indeed show that the union of the \( \ell' \)'-skewed cubes covers \( T^\downarrow \), the following theorem then guarantees the correctness of our approach:

**Theorem 2 ([5, Theorem 19]).** \( T \) is homeomorphic to \( \Gamma \) for \( \varepsilon < 0.06 \).

In particular, Amenta et al. [4] showed that the function mapping a point \( x \in T \) onto its closest point \( z \in \Gamma \) is a homeomorphism. Thus, we define \( \mu_1(z) = x \).

The following lemma shows that pushing (the vertices of) a triangle of \( T \) towards the interior of \( \Sigma \) does not distort the lengths of the resulting triangle too much:

**Lemma 18.** Let \( t \in T \) and \( s_1, s_2, s_3 \in S \) the corners of \( t \). For \( s, q \in \{s_1, s_2, s_3\} \) we have \( |s^\downarrow q^\downarrow| \leq (1 + 2 \cdot \delta) \cdot 3.2 \cdot \delta \cdot \min\{\text{afs}(s_1), \text{afs}(s_2), \text{afs}(s_3)\} \).

**Proof.** Let \( s, q \in \{s_1, s_2, s_3\} \) be chosen arbitrarily. W.l.o.g., we can assume that \( \text{afs}(s) \leq \text{afs}(q) \) holds. Let \( \ell_1 \) be the line that lies parallel to \( s^\downarrow \) such that \( q \in \ell_1 \). Furthermore, let \( c \) be the orthogonal projection of \( q^\downarrow \) on \( \ell_1 \) and let \( c' \) be the orthogonal projection of \( s^\downarrow \) onto \( \ell_1 \), see Figure 5(a).

In the following, we show that \( |s^\downarrow c| \leq 2.2 \cdot \delta \cdot \text{afs}(s) \) and \( |cq^\downarrow| \leq \delta \cdot \text{afs}(s) \) hold. The triangle inequality then implies \( |s^\downarrow q^\downarrow| \leq |s^\downarrow c| + |cq^\downarrow| \leq 3.2 \cdot \delta \cdot \text{afs}(s) \). As \( \text{afs}(s) \leq \text{afs}(q) \), we have \( |s^\downarrow q^\downarrow| \leq 3.2 \cdot \delta \cdot \min\{\text{afs}(s), \text{afs}(s)\} \). Also, \( \delta \cdot \text{afs}(s) = \max_{p \in S} \frac{\psi(s)}{\text{afs}(p)} \cdot \text{afs}(s) \geq \frac{\psi(s)}{\text{afs}(s)} \cdot \text{afs}(s) = \psi(s) = \max_{x \in \text{Vor}_S(s) \cap \Gamma} |s^\downarrow x| \), we have \( |sq| \leq 2 \cdot \delta \cdot \min\{\text{afs}(s), \text{afs}(q)\} \), and,
hence, using the same line of reasoning for all pairs of points in \( \{s_1, s_2, s_3\} \), we obtain \( |s^i q^i| \leq 3.2 \cdot \delta \cdot \min \{afs(s), afs(q)\} \leq (1 + 2 \cdot \delta) \cdot 3.2 \cdot \delta \cdot \min \{afs(s_1), afs(s_2), afs(s_3)\} \). This will conclude the proof.

We now consider the two inequalities mentioned above:

**Inequality 1** (\( |s^i c^i| \leq 2.2 \cdot \delta \cdot afs(s) \)): Let \( E_1 \) be the plane induced by \( s^i \), \( s \), and \( q \) and let \( E_2 \) the plane induced by \( q^i \), \( q \), and \( c \), see Figure 5. By rotating \( E_2 \) around \( \ell_1 \) and away from \( s^i \) until \( s \in E_2 \) (or, equivalently, until \( q^i \in E_1 \), i.e., until \( E_1 = E_2 \)) such that \( s \) and \( q^i \) lie on different sides w.r.t. \( \ell_1 \), \( |s^i q^i| \) is not decreased, see Figure 5(a). W.l.o.g. we thus can assume that \( E_1 = E_2 \).

![Figure 5: Worst-case construction for \( |s^i q^i| \) in the configuration of Lemma 18.](image)

For a worst-case configuration, we assume that \( s \) and \( q \) are at their maximal permissible distance; since both points are vertices of a Delaunay triangle, Lemma 1(2) implies \( |sq| = 2 \cdot \delta \cdot \min \{afs(s), afs(q)\} \). This in case, \( |s^i c^i| \) does not decrease, either. As \( afs(\cdot) \) is \( 1 \)-Lipschitz, the inequality \( |sq| \leq 2 \cdot \delta \cdot \min \{afs(s), afs(q)\} \) implies \( afs(q) \leq (1 + 2 \cdot \delta) \cdot afs(s) \). Since the construction of the pushed sample points (Algorithm 4) ensures \( |qq^i| = 15 \cdot \delta \cdot afs(q^i) \), this yields \( |qq^i| \leq (1 + 2 \cdot \delta) \cdot 15 \cdot \delta \cdot afs(s) \).

We define \( c' \) to be the point on \( \ell_1 \) closer to \( s^i \), i.e., the intersection of \( \ell_1 \) with a plane parallel to \( T_q \). Then, the previous considerations imply \( |cc'| \leq \sqrt{|sc|^2 + |cc|^2} \leq \sqrt{|sq|^2 + |cc|^2} \leq \sqrt{2^2 + (4.8 \cdot \delta) \cdot 2 + 2 \cdot \delta \cdot 15} \cdot \delta \cdot afs(s) \). So we have \( |s^i c^i| \leq \sqrt{|sc|^2 + |cc|^2} \leq \sqrt{|sq|^2 + |cc|^2} \leq \sqrt{2^2 + (4.8 \cdot \delta) \cdot 2 + 2 \cdot \delta \cdot 15} \cdot \delta \cdot afs(s) \). This is upper-bounded by \( 2.2 \cdot \delta \cdot afs(s) \) as claimed, assuming as usual that \( \varepsilon \) and, hence, \( \delta \in O(\varepsilon) \) are sufficiently small.

**Inequality 2** (\( |c q^i| \leq \delta \cdot afs(s) \)): Recall that \( q^i \in q p_q^- \) where \( p_q^- \) is the inner pole of \( q \). Let \( p_q^+ \) be the outer pole of \( q \). If we now rotate \( q^i \), \( p_q^- \), and \( p_q^+ \) on \( E_1 \) around \( q \) moving \( q^i \) away from \( s^i \) until the (polar) ball \( B_{|p_q^+ q|}(|p_q^- q|) \) contains \( s \) on its boundary (see Figure 5(b)), \( |s^i q^i| \) is not decreased. Define now \( T_q \) to be the plane orthogonal
Combining these two inequalities as discussed above concludes the proof.

Let Lemma 19.

$T$ statement) into the solid bounded by the surface. We start by showing that each triangle implying that a bridge edge does not penetrate "too deeply" (see Lemma 21 for a formal

This immediately implies that $T$ follow. In the following we show how to choose a segment $T$ has remained unchanged. Using a construction similar to the one used by Aichholzer et al. [1, Lemma 4.2], we can show that $T$, $|afs(q)|$ and $|afs(s)|$, see Figure 5(c). Combining this with $T$ $T$ derived above), we obtain $T$ $T$ as claimed.

Combining these two inequalities as discussed above concludes the proof.

Using the lemma we just proved allows us to show that the triangles in $T$ are properly contained in the union of the skewed cubes constructed for each of their vertices. This immediately implies that $T$ is properly contained in the union of all such cubes implying that a bridge edge does not penetrate “too deeply” (see Lemma 21 for a formal statement) into the solid bounded by the surface. We start by showing that each triangle in $T$ is contained in the union of certain balls centered at the triangle’s vertices:

**Lemma 19.** Let $t \in T$ and $s_1$, $s_2$, and $s_3$ the sample points that are the corners of $t$. Then $t^\perp \subset B_{2 \delta \cdot afs(s_1)}(s_1^\perp) \cup B_{2 \delta \cdot afs(s_2)}(s_2^\perp) \cup B_{2 \delta \cdot afs(s_3)}(s_3^\perp)$.

**Proof.** Let $x \in t^\perp$ be chosen arbitrarily. W.l.o.g., we can assume that $|s_1^\perp x| \leq |s_2^\perp x| \leq |s_3^\perp x|$ hold. In the following we show how to choose a segment $e \in \{s_1^\perp s_2^\perp, s_2^\perp s_3^\perp, s_3^\perp s_1^\perp\}$ such that $|x s_1^\perp| \leq \frac{|e|}{2 \cos(\pi/6)}$. Lemma 18 then will imply $|x s_1^\perp| \leq \frac{\delta - 3.39 \cdot \min\{afs(s_1), afs(s_2), afs(s_3)\}}{2 \cdot \cos(\pi/6)} \leq 2 \cdot \delta \cdot afs(s_1)$ as claimed.
In the following, we modify the configuration of $s_1^t$, $s_2^t$, and $s_3^t$ via two steps (see Figure 7) such that $|s_1^t x|$ is maintained and $|s_2^t x|$ and $|s_3^t x|$ are not increased. Finally, we show choose the edge $e$ and show $|x s_1^t| \leq \frac{|x|}{2 \cos(\pi/6)}$.

We first let $C_2$ be the disc with radius $x s_2^t$ centered at $x$, see Figure 7(a). By assumption, this implies $s_2^t \notin C_2$. Thus, pushing $s_2^t$ along $x s_2^t$ onto $C_2$ increases neither $|s_1^t s_2^t|$ nor $|s_1^t s_3^t|$. We now repeat this construction and push $s_2^t$ and $s_3^t$ towards $x$ onto the disc $C_1$ with radius $x s_1^t$ centered at $x$, see Figure 7(b). This neither increases $|x s_2^t|$ nor $|x s_3^t|$.

As we now can assume that $s_1^t$, $s_2^t$, $s_3^t$ lie on $C_1$, we have $|x s_1^t| = |x s_2^t| = |x s_3^t|$. W.l.o.g. we thus can assume that $\angle(s_2^t, s_1^t, s_3^t)$ is a smallest inner angle of $t$. This implies $\angle(s_2^t, s_1^t, s_3^t) \leq \frac{\pi}{3}$. W.l.o.g. we can assume that $\angle(x, s_1^t, s_2^t) \leq \angle(x, s_1^t, s_2^t)$. This implies $\angle(x, s_1^t, s_3^t) \leq \frac{\pi}{6}$ because the two modification steps (described above) maintain $x \in t$. This implies $|x s_1^t| \leq \frac{|x s_1^t|}{2 \cos(\pi/6)}$. This concludes the proof.

Lemma 20. For any $s' \in S^{sub}$, the union of all $s'$-skewed cubes covers $T^\downarrow$.

Proof. Lemma 19 ensures that each triangle $t^\downarrow \in T^\downarrow$ lies inside the union of the balls $B_{2.\delta_{off}(s_1)}(s_1^t)$, $B_{2.\delta_{off}(s_2)}(s_2^t)$, and $B_{2.\delta_{off}(s_3)}(s_3^t)$, where $s_1^t, s_2^t, s_3^t \in S^t$ are the pushed sample points induced by $t^\downarrow$. As these balls are surrounded by the $s'$-skewed cubes centered at $s_1^t$, $s_2^t$, and $s_3^t$ (Lemma 5), the lemma follows for any choice of $s' \in S^{sub}$.

We are now ready to show that a bridge edge $(s, q) \in S^{sub} \times S^{sub}$ does not penetrate too deep the (interior of the) solid $\Sigma$ bounded by $\Gamma$. Theorem 2 guarantees that there is a homeomorphism $\mu_1 : \Gamma \rightarrow T$. We formalize the space $\Delta$ between $T^\delta$ and $\Gamma$ as follows: For each $t = \triangle(s_1, s_2, s_3) \in T$ with $s_1, s_2, s_3 \in S$ and $\zeta \in [0, 1]$, we define $t^\zeta := \triangle(s_1 + \zeta(s_1 - s_1), s_2 + \zeta(s_2 - s_2), s_3 + \zeta(s_3 - s_3))$. Also, for $x \in \Gamma$ and $\zeta \in [0, 1]$, we define $x^\zeta := x + \zeta(\mu_1(x) - x)$. Finally, we denote $\Delta := \left(\bigcup_{t \in \Gamma, \zeta \in [0, 1]} t^\zeta\right) \cup \left(\bigcup_{x \in \Gamma, \zeta \in [0, 1]} x^\zeta\right)$.

Assume now that there was some $x \in \overline{s q} \cap \Sigma$ not in $\Delta$. Theorem 2 and the construction of $T^\downarrow$ then would imply the existence of some intersection point $y$ of $\overline{s q}$ and some $t \in T^\delta$. Lemma 20 would then imply $y$ to lie in the interior of the skewed cubes used during
the construction of the visibility edge between $s$ and $q$—a contradiction to the correctness of the space-sweep algorithm that does not allow the visibility edges constructed to intersect the interior of a skewed cube.

More formally, by our construction above, there is a continuous and surjective function $\mu_2 : T \to T^i$. Thus, $\mu := \mu_2 \circ \mu_1$ is surjective and continuous. This construction of $\mu$ implies that $T^i$ has the same genus as $\Gamma$, i.e., has no extra holes.\footnote{Note that we cannot guarantee that the triangles from $T^i$ are intersection-free.} Using elementary manipulations, we can show that $\Delta$ is not “too thick”:

**Lemma 21.** For each $x \in \Delta$, there is an $s_x \in S^{sub}$ such that $|xs_x| \leq 18 \cdot \delta \cdot afs(s_x)$.

**Proof.** Fix any point $x \in \Delta$. By the definition of $\Delta$, there are $z \in \Gamma$ and $\zeta \in [0,1]$ such that $x = z + \zeta(\mu_1(z) - z)$ or there are $t \in T$ and $\zeta \in [0,1]$ such that $x \in t\zeta$. We consider both cases separately:

**Case 1 ($x = z + \zeta(\mu_1(z) - z)$):** Let $t \in T$ be a triangle such that $\mu_1^{-1}(x) \in t$ (as $\mu_1^{-1}(x)$ may lie on an edge of $T$, $t$ needs not be defined uniquely) and define $s_x$ to be a corner of $t$ closest to $x$. As $t$ is a Delaunay triangle, Lemma 1 implies $|\mu_1(z)s_x| \leq \psi(s_x) \leq \delta \cdot afs(s_x)$. Furthermore, as (1) $s_x \in \Gamma$ and (2) $z$ is by definition of $\mu_1$ the closest point from $\Gamma$ to $\mu_1(z)$, we obtain $|\mu_1(z)z| \leq |\mu_1(z)s_x|$ implying $|\mu_1(z)z| \leq \delta \cdot afs(s_x)$. As $x$ lies on the segment between $z$ and $\mu_1(z)$, the triangle inequality implies $|xs_x| \leq |x\mu_1(z)| + |\mu_1(z)s_x| \leq 2 \cdot \delta \cdot afs(s_x)$.

**Case 2 ($x \in t\zeta$):** The approach of Lemma 19 also applies to $t\zeta$ and its three corners $p_1$, $p_2$, and $p_3$. Let $u \in \{p_1, p_2, p_3\}$ be the corner that lies closest to $x$ and $s_x \in \{s_1, s_2, s_3\}$ the corresponding sample point. Thus, $|xu| \leq 2 \cdot \delta \cdot afs(s_x)$. By the construction used in Algorithm 4, we have $|s_xu| \leq 15 \cdot \delta \cdot afs(s_x)$. The triangle inequality yields $|s_xx| \leq |s_xu| + |ux| \leq 17 \cdot \delta \cdot afs(s_x)$.

If $s_x \in S^{sub}$, the lemma follows by setting $s_x := s_x'$. If $s_x \notin S^{sub}$, let $s_x \in S^{sub}$ be the sample point that caused $s_x'$ to not be included in $S^{sub}$; this yields $|s_xs_x'| \leq 0.1 \cdot \delta \cdot afs(s_x)$ (see Algorithm 2) and we obtain $|s_xx| \leq 17 \cdot \delta \cdot afs(s_x) + 0.1 \cdot \delta \cdot afs(s_x)$. As $afs(\cdot)$ is 1-Lipschitz, we have $afs(s_x') \leq (1 + 0.1 \cdot \delta) \cdot afs(s_x)$. This implies $|s_xx| \leq 17 \cdot (1 + 0.1 \cdot \delta) \cdot \delta \cdot afs(s_x) \leq 18 \cdot \delta \cdot afs(s_x)$. \hfill \Box

As $\mu$ is surjective and continuous, $\Sigma \setminus \Delta$ is bounded by a subset of $T^i$. Combining this with Lemmas 21 and 20 implies that $\overline{\Sigma}$ does not penetrate $\Sigma$ “too deeply” as formalized in Lemma 8 (BE2):

**Lemma 22.** [Lemma 8 (BE2)]: For each $x \in \overline{\Sigma} \cap \Sigma$ there is an $s_x \in S^{sub}$ with $|xs_x| \leq 18 \cdot \delta \cdot afs(s_x)$.

**Proof.** Assume that there is some $x \in \overline{\Sigma} \cap \Sigma$ such that there is no $s_x \in S^{sub}$ with $|xs_x| \leq 18 \cdot \delta \cdot afs(s_x)$. The contrapositive of Lemma 21 implies $x \notin \Sigma \setminus \Delta$. Thus, there exists some $y \in \overline{\Sigma} \cap T^i$. This implies for all $\pi \in \Pi_s$, there is some cube $c \in C(\pi, s)$ such that $\overline{\Sigma} \cap c \neq \emptyset$.\hfill \Box
Analogously, we obtain for all $\pi \in \Pi_q$, there is some cube $c \in \mathbb{C}(\pi, q)$ such that $\overline{sq} \cap c \neq \emptyset$. As $(s, q) \in E_{\text{bri}}$, this is a contradiction to the correctness of the space-sweep algorithm. \qed

Now we are ready to prove Lemma 17:

**Lemma 17:** For $(s, q) \in E_{\text{bri}} \subseteq S_{\text{sub}} \times S_{\text{sub}}$, we have $L^*_s (s, q) \leq (1 + \mathcal{O}(\sqrt{\varepsilon})) \cdot |sq|$.

**Proof.** By construction of $E_{\text{bri}}$, we have $|sq| > \frac{1}{\delta} \cdot \sqrt{\varepsilon} \cdot \min \{afs(s), afs(q)\}$.

We follow the same general approach as in the proof of Lemma 16: For $a, b \in \mathbb{R}^q$, we iteratively construct two sample points $s_a, s_b \in S_{\text{sub}}$ and a curve $\gamma_{s_a s_b}$ connecting $s_a$ and $s_b$ inside the free space $\Lambda$ such that $|\gamma_{s_a s_b}| \leq (1 + \mathcal{O}(\sqrt{\varepsilon})) \cdot |ab|$. We start with $s_a := s$ and $a := s$ and, just as in the proof of Lemma 16, repeat the steps discussed below updating $a$ with the previous value of $b$ and $s_a$ with the previous value of $s_b$ until an iteration reaches $s_b = q$. We start by defining $\gamma_{sq} \in \Lambda$ to be the curve between $s$ and $q$ that is given by the concatenation of the curves mentioned above and as constructed below. By the above argument we finally obtain $|\gamma_{sq}| \leq (1 + \mathcal{O}(\sqrt{\varepsilon})) \cdot |sq|$. As $L^*_s (s, q) \leq |\gamma_{sq}|$, we get $L^*_s (s, q) \leq (1 + \mathcal{O}(\sqrt{\varepsilon})) \cdot |sq|$.

The steps performed during the iterative construction are as follows: For the first subcurve we set $a := s$ and $s_a := s$. In each iteration, we define $x$ to be a point on $aq$ such that $|ax| = \frac{1}{b} \cdot \sqrt{\varepsilon} \cdot \min \{afs(a), afs(x)\}$ (such a point exists in the first iteration as we assumed that $|sq| > \frac{1}{\delta} \cdot \sqrt{\varepsilon} \cdot \min \{afs(s), afs(q)\}$). A standard argument using the Lipschitz continuity of $afs(\cdot)$ shows that in this case $afs(s) \in \Theta(afs(q))$.

To proceed with the proof, we distinguish whether or not $x \in \Sigma$ holds:

**Case 1 ($x \in \Sigma$):** We set $b := x$. Lemma 8 (BE2) implies that there is some $s_b \in S_{\text{sub}}$ such that $|bs_b| \leq 18 \cdot \delta \cdot afs(s_b)$. In particular, this implies $|as_b| \leq \frac{1}{3} \cdot \sqrt{\varepsilon} \cdot \min \{afs(a), afs(s_b)\}$. Hence, Lemma 3 (LE1) implies $L^*_s (a, s_b) \leq (1 + \mathcal{O}(\sqrt{\varepsilon})) \cdot |as_b|$. If we can show that $|as_b| \leq (1 - \mathcal{O}(\sqrt{\varepsilon}))^{-1} \cdot |ab|$, we obtain $L^*_s (a, s_b) \leq (1 + \mathcal{O}(\sqrt{\varepsilon})) \cdot |as_b| \leq (1 + \mathcal{O}(\sqrt{\varepsilon})) \cdot (1 - \mathcal{O}(\sqrt{\varepsilon}))^{-1} \cdot |ab| \leq (1 + \mathcal{O}(\sqrt{\varepsilon})) \cdot |ab|$ as claimed.

To relate $|as_b|$ and $|ab|$ in the desired way, we rearrange the triangle inequality to yield $|as_b| - |s_b| \leq |ab|$ and observed that relating $|bs_b| \leq \mathcal{O}(\sqrt{\varepsilon}) \cdot |as_b|$ would then imply $(1 - \mathcal{O}(\sqrt{\varepsilon})) \cdot |as_b| \leq |ab|$ as needed. Hence, we first bound $|ab| = \frac{1}{b} \cdot \sqrt{\varepsilon} \cdot \min \{afs(a), afs(b)\}$ and conclude using the Lipschitz continuity of $afs(\cdot)$ that $afs(b) \geq (1 - \frac{1}{b} \cdot \sqrt{\varepsilon} \cdot \min \{afs(a), afs(b)\})$.

Similarly, we derive $afs(b) \geq (1 - \mathcal{O}(\delta) \cdot afs(s_b))$. Combining these inequalities yields $|ab| = \frac{1}{b} \cdot \sqrt{\varepsilon} \cdot \min \{afs(a), afs(b)\} \geq \Theta(\sqrt{\varepsilon}) \cdot afs(b) \geq \Theta(\sqrt{\varepsilon}) \cdot (1 - \mathcal{O}(\delta)) \cdot afs(s_b) \geq \Theta(\sqrt{\varepsilon}) \cdot afs(s_b)$.

By Lemma 8 (BE2), we know that $|bs_b| \leq \mathcal{O}(\delta) \cdot afs(s_b)$, i.e., $|bs_b| \leq \mathcal{O}(\sqrt{\varepsilon}) \cdot |ab|$. Plugging this into the triangle inequality $|as_b| \geq |ab| - |bs_b|$ yields $|as_b| \geq (1 - \mathcal{O}(\sqrt{\varepsilon})) \cdot |ab| \geq (1 - \mathcal{O}(\sqrt{\varepsilon})) \cdot \Theta(\sqrt{\varepsilon}) \cdot afs(s_b) \geq \Theta(\sqrt{\varepsilon}) \cdot afs(s_b)$. Starting again from $|bs_b| \leq \mathcal{O}(\delta) \cdot afs(s_b)$, we then obtain $|bs_b| \leq \mathcal{O}(\sqrt{\varepsilon}) \cdot \Theta(\sqrt{\varepsilon}) \cdot afs(s_b) \leq \mathcal{O}(\sqrt{\varepsilon}) \cdot |as_b|$. Completing the reasoning as indicated above with $(1 - \mathcal{O}(\sqrt{\varepsilon}))^{-1} = (1 + \mathcal{O}(\sqrt{\varepsilon}))$, we obtain $L^*_s (a, s_b) \leq (1 + \mathcal{O}(\sqrt{\varepsilon})) \cdot |ab|$ as claimed.

**Case 2 ($x \notin \Sigma$):** The main idea in handling this case is to ensure that we proceed along $sq$ until we hit a point on $\Gamma$; this in turn will guarantee that there is a sample point nearby.
As $x \notin \Sigma$, we actually need to find two such points: the point $y$ where $\text{sq}$ exits $\Sigma$ for the last time before passing through $x$ and the point $z$ where $\text{sq}$ enters $\Sigma$ immediately after having passed through $x$. More formally, let $y \in \Gamma \cap ax$ (note that $ax \subseteq \text{sq}$) such that $yx \cap \Sigma = \{y\}$. Lemma 8 (BE2) implies that there is an $s_y \in S^{\text{sub}}$ such that $|ys_y| \leq 18 \cdot \delta \cdot \text{afs}(s_y)$. We distinguish whether or not $|ay| \geq \frac{1}{6} \cdot \sqrt{\delta} \cdot \min\{\text{afs}(a), \text{afs}(y)\}$ holds.

**Case 2.1** ($|ay| \geq \frac{1}{6} \cdot \sqrt{\delta} \cdot \min\{\text{afs}(a), \text{afs}(y)\}$): We set $b := y$ and $s_b := s_y$. An argument analogous to the one used in Case 1 (above) bounds $|ab|$ The triangle inequality then yields $L_1^+ (a, s_b) \leq (1 + O(\sqrt{\delta})) \cdot |ab|$.

**Case 2.2** ($|ay| < \frac{1}{6} \cdot \sqrt{\delta} \cdot \min\{\text{afs}(a), \text{afs}(y)\}$): By applying that $\text{afs}(\cdot)$ is 1-Lipschitz continuous, we can show $|as_y| \leq \frac{1}{3} \cdot \sqrt{\delta} \cdot \min\{\text{afs}(a), \text{afs}(s_y)\}$. Thus, Lemma 3 (LE1) implies $L_1^+ (a, s_y) \leq (1 + O(\sqrt{\delta})) \cdot |as_y|$. To construct the second point needed in this case, i.e., the point where $\text{sq}$ enters $\Sigma$ immediately after having passed through $x$, let $z \in \Sigma \cap \text{sq}$ such that $xz \cap \Sigma = \{z\}$ (note that $q$ is a sample point, hence, $z$ is well-defined). Lemma 8 (BE2) implies that there is an $s_z \in S^{\text{sub}}$ such that $|zs_z| \leq 18 \cdot \delta \cdot \text{afs}(s_z)$; we set $b := z$ and $s_b := s_z$. By applying 1-Lipschitz continuity of $\text{afs}(\cdot)$ we can show $|yb| \geq O(\sqrt{\delta}) \cdot \max\{\text{afs}(y), \text{afs}(b)\}$. This implies $|ys_y|, |bs_b| \leq O(\sqrt{\delta}) \cdot |yz|$. Lemma 25 implies $L_1^+ (y, s_y) \leq (1 + O(\sqrt{\delta})) \cdot |ys_y|$ and $L_1^+ (b, s_b) \leq (1 + O(\sqrt{\delta})) \cdot |bs_b|$. Furthermore, we have $L_1^+ (y, b) = |yb|$ because $yb \subseteq \Lambda$. Thus, we can upper bound

\[
L_1^+ (a, s_b) \leq L_1^+ (a, s_y) + L_1^+ (s_y, s_b) \\
\leq (1 + O(\sqrt{\delta})) \cdot |as_y| + L_1^+ (s_y, y) + L_1^+ (y, b) + L_1^+ (b, s_b) \\
\leq (1 + O(\sqrt{\delta})) \cdot |as_y| + (1 + O(\sqrt{\delta})) \cdot |s_yy| + |yb| + (1 + O(\sqrt{\delta})) \cdot |bs_b| \\
\leq (1 + O(\sqrt{\delta})) \cdot (1 + O(\sqrt{\delta})) \cdot |ay| + (1 + O(\sqrt{\delta})) \cdot O(\sqrt{\delta}) \cdot |yb| + |yb| \\
\leq (1 + O(\sqrt{\delta})) \cdot |ab|
\]

As we have shown that $L_1^+ (a, s_y) \leq (1 + O(\sqrt{\delta})) \cdot |ab|$ holds in all cases, we can iteratively construct an piecewise and, hence, global approximation of $\text{sq}$ using subcurves of the type $\gamma_{s_a, s_b}$. This concludes the proof.

Combining Lemmas 3 (LE1) and 17 yields the lower bound for all edges, i.e., both local edges and bridge edges are $(1 + O(\sqrt{\delta}))$-approximations of the respective geodesics in the sense that their lengths are not too short.

### 3.2 Upper-Bounding the Edge Length

To ensure $L (s_1, s_2) \leq (1 + O(\sqrt{\delta})) \cdot L_1^+ (s_1, s_2)$ for all $s_1, s_2 \in S$, we again distinguish whether $|s_1s_2| \leq \frac{1}{3} \cdot \sqrt{\delta} \cdot \min\{\text{afs}(s_1), \text{afs}(s_2)\}$ holds. If this is the case, we have $L (s_1, s_2) = |s_1s_2|$, which is trivially upper-bounded by $(1 + O(\sqrt{\delta})) \cdot L_1^+ (s_1, s_2)$.

Lemma 4 yields $|s_1\nu_{s_1}| \leq O(\delta) \cdot \text{afs}(\nu_{s_1})$ and $|s_2\nu_{s_2}| \leq O(\delta) \cdot \text{afs}(\nu_{s_2})$ (recall that, for a sample point $s \in S$, $\nu_s$ denotes the nearest neighbor of $s$ in $S^{\text{sub}}$; see Algorithm 6). Lemma 16
implies $L^*_t(s_1, ν_1) \leq (1 + O(√ε)) \cdot |s_1 ν_1|$ and $L^*_t(s_2, ν_2) \leq (1 + O(√ε)) \cdot |s_2 ν_2|$. Furthermore, by 1-Lipschitz continuity, $|s_1 ν_2| > \frac{1}{3} \sqrt{δ} \cdot \min\{αfs(s_1), αfs(s_2)\}$ implies $|s_1 ν_2| ≥ \frac{1}{1-2} \cdot \frac{1}{3} \sqrt{δ} \cdot \max\{αfs(s_1), αfs(s_2)\}$ which is lower-bounded by $Ω(√δ) \cdot αfs(s_1)$ and $Ω(√δ) \cdot αfs(s_2)$. Thus, $L^*_t(s_1, ν_1) ≤ O(\frac{δ}{Ω(1/√δ)}) \cdot |s_1 ν_1|$, and $L^*_t(s_2, ν_2) ≤ O(\frac{δ}{Ω(1/√δ)}) \cdot |s_1 ν_2|$. The triangle inequality implies $L^*_t(s_1, ν_2) ≤ L^*_t(s_1, ν_1) + L^*_t(ν_1, ν_2) + L^*_t(ν_1, s_1) ≤ (1 + O(√ε)) \cdot L^*_t(ν_1, s_1, ν_2)$. As $ν_1, ν_2 \in S^{sub}$, we will be able show the required upper bound for $L^*_t(s_1, s_2)$ by applying Lemma 8 (BE2). In the remainder of this section, we will prove the following lemma:

**Lemma 23.** For all $s_1, s_2 \in S^{sub}$, $L^*_t(s_1, s_2) ≤ (1 + O(√ε)) \cdot L^*_t(s_1, s_2)$.

The main idea is as follows: To ensure $L^*_t(s_1, s_2) ≤ (1 + O(√ε)) \cdot L^*_t(s_1, s_2)$ for all $s_1, s_2 \in S^{sub}$, we consider a geodesic $γ \subset Λ$ between $s_1$ and $s_2$. We show that there is a path $φ$ in the graph $G = (S^{sub}, E_{loc} \cup E_{brn})$ constructed in Algorithm 1 such that $φ$ connects $s_1$ and $s_2$ and $|φ| ≤ (1 + O(√ε)) \cdot |γ|$ holds. In particular, $γ$ will be constructed from curves on $Γ$ and certain segments that lie in $Λ$ and connect two points from $Γ$. To guarantee $|φ| ≤ (1 + O(√ε)) \cdot |γ|$ we need to ensure that both types of parts are approximated appropriately: subcurves on $Γ$ (which then can be approximated by edges in $E_{loc}$), and segments in $Λ$ connecting two points on $Γ$, see Lemma 26. Finally, we combine the properties of local edges (Lemma 3), Lemma 26, and some rather technical auxiliary lemmas to construct $φ \subset G$ such that $|φ| ≤ (1 + O(√ε)) \cdot |γ|$, see Lemma 23.

As the path $φ$ will be constructed to connect two sample points in $S^{sub}$, we set up the proofs for the following two technical lemmas by fixing two sample points $s, s' \in S^{sub}$. Furthermore, we fix a pyramid $π \in Π_r$ and a cube $c \in C(π, s')$. We first show that pushing all skewed cubes towards the inner poles indeed pushed all cubes are pushed “sufficiently far away” from $Γ$, i.e., towards the interior of $Σ$.

**Lemma 24.** For any point $z$ in the (full-dimensional) cube $c$ and any point $x \in Γ$, we have $|xz| ≥ 9 \cdot δ \cdot \max\{αfs(x), αfs(z)\}$.

**Proof.** Fix a point $z$ in $c$ and a point $x \in Γ$. Let $s \in S$ be the sample point that induced the construction of $c$. In a first step, we will show that $|xz| ≥ 11 \cdot δ \cdot αfs(s) - 1 \cdot \frac{17δ}{1 - 1.17δ} \cdot αfs(x)$ holds. Based on that we show $|xz| ≥ 9 \cdot δ \cdot αfs(x)$ and, finally, $|xz| ≥ 9 \cdot δ \cdot αfs(s)$ separately.

**Step 1** ($|xz| ≥ 11 \cdot δ \cdot αfs(s) - 1 \cdot \frac{17δ}{1 - 1.17δ} \cdot αfs(x)$): Let $p$ be the inner pole point of $s$. By Lemma 4, there is at least one sample point $s \in S^{sub}$ at most $1.17 \cdot δ \cdot αfs(\tilde{s})$ away from $x$. The triangle inequality yields $|xp| ≥ |p\tilde{s}| + |\tilde{s}x|$. As $p$ is a pole, i.e., as $B_{|ps|}(p)$ does not contain any sample point, we conclude that $|xp| ≥ |p\tilde{s}| - 1 \cdot \frac{17δ}{1 - 1.17δ} \cdot αfs(x)$ (using that, by the 1-Lipschitz continuity, we have $αfs(\tilde{s}) ≤ 1 \cdot \frac{17δ}{1 - 1.17δ} \cdot αfs(x)$).

**Lemma 5,** on the other hand, implies $|zs| ≤ 3.82 \cdot δ \cdot αfs(s)$. Furthermore, we have $|ss'| = 15 \cdot δ \cdot αfs(s)$. As $s' \in \overline{sp}$ (recall that this point was constructed by pushing $s$ along $\overline{sp}$), we can lower-bound $|ys'| ≥ 15 \cdot δ \cdot αfs(s)$ for all $y \in \partial B_{|ys|}(p)$. We distinguish whether or not $x \in B_{|ys|}(p)$ holds:

**Case 1:** $x \notin B_{|ys|}(p)$: The triangle inequality implies $|xz| + |zs'| ≥ |xz'|$ or, equivalently, $|xz| ≥ |xz'| - |zs'|$. Thus $|xz| ≥ 15 \cdot δ \cdot αfs(s) - 3.82 \cdot δ \cdot αfs(s) > 11 \cdot δ \cdot αfs(s)$. 
Case 2: $x \in B_{|p|}(p)$: Let $y \in \partial B_{|p|}(p)$ be a point in $\partial B_{|p|}(p)$ closest to $x$. This implies $|xy| \leq \frac{11}{1-1.1773}\cdot af(s).$ By the above we obtain $|xz| \geq 11\cdot \delta \cdot af(s) - \frac{11}{1-1.1773}\cdot af(s).$

**Step 2** ($|xz| \geq 9.5\cdot \delta \cdot af(s)$): For the sake of contradiction, we assume $|xz| < 9.5\cdot \delta \cdot af(s)$. The triangle inequality implies $|zs| \leq |xz| + |zs| \leq |xz| + |s| + |s| \leq 9.5\cdot \delta \cdot af(s) + 3.83\cdot \delta \cdot af(s) + 15\cdot \delta \cdot af(s) \leq 9.5\cdot \delta \cdot af(s) + 19\cdot \delta \cdot af(s).$ As $af(s)$ is 1-Lipschitz, we obtain $af(s) \leq af(s) + 9.5\cdot \delta \cdot af(s) + 19\cdot \delta \cdot af(s).$ This is equivalent to $af(s) \leq \frac{1+19\cdot \delta}{1-1.1773}\cdot af(s)$ which implies $|xz| < 9.5\cdot \frac{1}{1+19\cdot \delta}\cdot \delta \cdot af(s) < 9.7\cdot \delta \cdot af(s).$ Above we showed $|xz| \geq 11\cdot \delta \cdot af(s) - \frac{11}{1-1.1773}\cdot af(s),$ Hence, we obtain $|xz| \geq 11\cdot \delta \cdot af(s) - \frac{11}{1-1.1773}\cdot \delta \cdot af(s).$ This is a contradiction.

**Step 3** ($|xz| > 9\cdot \delta \cdot af(s)$): For the sake of contradiction we assume $|xz| \leq 9\cdot \delta \cdot af(s)$. By applying a similar argument as in the previous case we obtain $af(s) \leq \frac{1+9.5\cdot \delta}{1-1.1773}\cdot af(s)$ which implies $|xz| \leq 9\cdot \frac{1}{1+9.5\cdot \delta}\cdot \delta \cdot af(s) < 9.5\cdot \delta \cdot af(s).$ This is a contradiction to the previous case.

Hence, we have shown $|xz| > 9\cdot \delta \cdot \max\{af(s), af(z)\}$ as claimed. □

An immediate consequence of Lemma 24 is that $c \subset \Sigma$ for each $s' \in S^{sub}, \pi \in P_s,$ and $c \in C(\pi, s')$. Thus we have:

**Corollary 4.** For each $(s, s') \in S^{sub} \times S^{sub}$ we have $s \in V(s')$ if $(s, s') \cap \Sigma = \emptyset.$

**Corollary 4** means that the entire visibility neighborhood $V(s')$ of a sample point $s' \in S^{sub}$ contains all visibility edges towards points from $S^{sub}$ w.r.t. $\Sigma$. Building upon Lemma 24, we prove the following lemma which resembles Lemma 3 (LE1) in the case that only one of the two points in question is a sample point. In contrast to Lemma 3 (LE1) the two points now need be located much closer to each other in order to allow for a good approximation. In contrast to Lemma 12, on the other hand, this distance now depends on $\delta$ instead of $\sqrt{\varepsilon}$; this allows the following lemma to be used in the (constructive) context of our algorithm where $\delta$ but not $\sqrt{\varepsilon}$ is available.

**Lemma 25.** For $s \in S^{sub}$ and $x \in \Gamma$ with $|sx| \leq 1.17\cdot \delta \cdot af(s)$ we have $L^*_s(s, x) \leq (1 + O(\sqrt{\varepsilon}))\cdot |sx|.$

**Proof.** If $|sx| \leq \sqrt{\varepsilon} \cdot \min\{|lfs(s), lfs(x)|\},$ Lemma 13 implies $L^*_s(s, x) \leq (1 + O(\varepsilon))\cdot |sx|.$ Thus, we assume $|sx| > \sqrt{\varepsilon} \cdot \min\{|lfs(s), lfs(x)|\}$. Let $s_x \in S$ be a closest sample point to $x$. As $S$ is an $\varepsilon$-sample, we have $|sx_x| \leq \varepsilon lfs(x)$. Hence, combining $|sx| \leq \sqrt{\varepsilon} \cdot \min\{|lfs(s), lfs(x)|\}$ and $|sx_x| \leq \varepsilon lfs(x)$ yields $s_x |sx| \leq O(\sqrt{\varepsilon}) |sx|$. The triangle inequality implies $|ss_x| \leq |sx| + |sx_x| \leq (1 + O(\sqrt{\varepsilon})) |sx|$

Additionally, $|ss_x| \leq \varepsilon \cdot lfs(x)$ implies $|ss_x| \leq \frac{\varepsilon}{1-\varepsilon} \cdot \min\{|lfs(s_x), lfs(x)|\}$. Hence, Lemma 13 implies $L^*_s(s_x, x) \leq (1 + O(\varepsilon)) \cdot |ss_x|.$

By the definition of the control function $\psi(\cdot)$, we know that $\psi(s_x) \geq |sx_x|$. Hence, the triangle inequality implies $|ss_x| \leq |sx| + |sx_x| \leq 1.17 \cdot \delta \cdot af(s) + \psi(s_x) \leq 1.17 \cdot \delta \cdot af(s) + \psi(s_x) \leq 1.17 \cdot \delta \cdot af(s)$.
\[
\psi(s_x) \cdot \text{afs}(s_x) \leq 1.17 \cdot \delta \cdot \text{afs}(s) + \max_{s' \in S} \psi(s') \cdot \text{afs}(s_x) = 1.17 \cdot \delta \cdot \text{afs}(s) + \delta \cdot \text{afs}(s_x).
\]

As \(\psi(\cdot)\) is 1-Lipschitz, we get \(\text{afs}(s) \leq \text{afs}(s_x) \leq 1.17 \cdot \delta \cdot \text{afs}(s) + \delta \cdot \text{afs}(s_x)\) or, equivalently, \(\text{afs}(s) \leq \frac{1+\delta}{1.17} \cdot \text{afs}(s_x)\) and \(\text{afs}(s_x) \leq \text{afs}(s) + 1.17 \cdot \delta \cdot \text{afs}(s) + \delta \cdot \text{afs}(s_x)\). This is equivalent to \(\text{afs}(s_x) \leq \frac{1+1.17\delta}{1.17} \cdot \text{afs}(s)\). This implies \(|ss_x| \leq 1.17 \cdot \delta \cdot \text{afs}(s) + 1.17 \cdot \delta \cdot \text{afs}(s_x) \leq \frac{3}{3} \cdot \delta \cdot \text{afs}(s)\) and \(|ss_x| \leq 1.17 \cdot \delta \cdot \frac{1+1.17\delta}{1.17} \cdot \text{afs}(s_x) + \delta \cdot \text{afs}(s_x) \leq \frac{2}{3} \cdot \delta \cdot \text{afs}(s_x)\). Hence, \(|ss_x| \leq \frac{1}{3} \cdot \delta \cdot \min\{\text{afs}(s), \text{afs}(s_x)\}\). By Lemma 3 (LE1) we obtain \(L^*_z(s, s_x) \leq (1 + O(\sqrt{\varepsilon})) \cdot |ss_x|\).

Putting everything together leads to \(L^*_z(s, s_x) \leq L^*_z(s, s_y) + L^*_z(s_y, x) \leq (1 + O(\varepsilon)) \cdot |ss_x| + (1 + O(\varepsilon)) \cdot |ss_y| + (1 + O(\varepsilon)) \cdot \text{afs}(s_y) \leq (1 + O(\sqrt{\varepsilon})) \cdot |ss_x| \leq (1 + O(\sqrt{\varepsilon})) \cdot |ss| \leq (1 + O(\sqrt{\varepsilon})) \cdot |sx| \leq (1 + O(\sqrt{\varepsilon})) \cdot |sx|.\) This concludes the proof.

\[\square\]

In the following lemma, we consider the case of two arbitrary points \(x\) and \(y\) on the manifold that are directly visible from each other. We show that there is a graph edge, i.e., a local edge or a bridge edge, connecting two sample points \(s_x\) and \(s_y\) close to these points or that there is at least one other sample point that lies in the same cone with apex at \(s_x\) as \(s_y\) and vice versa. In the first case, the length of the edge approximates the geodesic distance between the two points in question. The configuration of the second case will be used to incrementally construct the desired approximation, see the proof of Lemma 23 given after the next lemma.

**Lemma 26.** Let \(x, y \in \Gamma\) such that \(xy \cap \Sigma = \emptyset\). Furthermore, let \(s_x\) be a sample point in \(S^{\text{sub}}\) closest to \(x\) and let \(s_y\) be a sample point in \(S^{\text{sub}}\) closest to \(y\). Then, \((s_x, s_y) \in E_{\text{loc}} \cup E_{\text{bri}}\) holds or there exists a sample point \(s'_y\) in the approximate visibility neighborhood \(A(s_x)\) and a sample point \(s'_x\) in the approximate visibility neighborhood \(A(s_y)\) such that \(s'_y\) lies in the same cone \(C_x \in C(s_x)\) as \(s_y\) and such that \(s'_x\) lies in the same cone \(C_y \in C(s_y)\) as \(s_x\).

**Proof.** If \(|ss_x| \leq \frac{1}{3} \cdot \sqrt{\delta} \cdot \min\{\text{afs}(s_x), \text{afs}(s_y)\}\), the algorithm constructs \((s_x, s_y)\) as a local edge, hence, \((s_x, s_y) \in E_{\text{loc}}\). Thus, we can assume w.l.o.g. that \(|ss_y| > \frac{1}{3} \cdot \sqrt{\delta} \cdot \min\{\text{afs}(s_x), \text{afs}(s_y)\}\) holds. In the following, we show that there is an \(s'_y \in A(s_x)\) that lies in \(C \in C(s_x)\) where \(C\) is chosen such that \(s_y \in C\). A symmetric argument implies the corresponding statement for \(s_y\).

Let \(l\) be the line through \(s_x\) and parallel to \(xy\) and let \(z\) be the point on \(l\) with \(|sz| = 4 \cdot \sqrt{\delta} \cdot \text{afs}(s_x)\). Let \(E_z\) be the plane orthogonal to \(l\) such that \(z \in E_z\), and let \(E_{s_y}\) be the plane orthogonal to \(l\) such that \(s_y \in E_{s_y}\), see Figure 8. Furthermore, let \(C \in C(s_x)\) with \(\max_{u \in C} \angle (us_x, ss_y) \leq \frac{1}{2} \cdot \sqrt{\delta}\). We denote the part of \(C\) that lies to the left of \(E_z\) by \(C_z\) and the part of \(C\) that lies between \(E_z\) and \(E_{s_y}\) by \(C_{s_y}\).

If there is an \(s'_y \in C \cap A(s_x) \setminus \{s_y\}\) we have shown that the second part of the statement hold and can conclude the proof. Thus, we assume that there is no \(s'_y \in C \cap A(s_x) \setminus \{s_y\}\). In the following, we show that \(C \cap A(s_x) \setminus \{s_y\} = \emptyset\) implies \(s_y \in A(s_x)\). This is equivalent to \((s_x, s_y) \in B\) which in turn concludes the proof as well.

Fix any cube \(c \in C(\pi_s, s_x)\) for an arbitrary pyramid \(\pi \in \Pi_{s_x}\), see Definition 2. In the following, we show that both \(c \cap C_z = \emptyset\) and \(c \cap C_{s_y} \cap \text{afs}(s_y) = \emptyset\) holds. As \(c\) was
chosen arbitrarily, we then know that $s_x s_y$ does not intersect any cube, hence, by definition, $s_y \in A(s_x)$.

To show that $c \cap C_x = \emptyset$ hold, we assume for the sake of contradiction that there is a point $c_1$ in the cube $c$ such that $c_1 \in C_x$. We show that this cube then would lie too close to a point on $xy$, i.e., to a point in the free space, hence contradicting Lemma 24. More formally, let $E_{c_1}$ be the plane that lies orthogonal to $\ell$ such that $c_1 \in E_{c_1}$, $c'_1$ the orthogonal projection of $c_1$ onto $xy$, and $c''_1 = E_{c_1} \cap \ell$, see Figure 8.

We have $|s_x c''_1| \leq |s_x z| = 4\sqrt{\delta} \cdot afs(s_x)$. This implies $afs(s_x) \geq \left(1 - 4\sqrt{\delta}\right) \cdot afs(c''_1)$. Furthermore, we have $|s_x x| \leq \delta \cdot afs(s_x)$ implying $|c''_1 c'_1| \leq \delta \cdot afs(s_x)$. Hence, we obtain $|c'_1 c''_1| \leq \frac{\delta}{1 - 4\sqrt{\delta}} \cdot afs(c''_1) \leq 1.01 afs(c''_1)$. Thus, we have $afs(c'_1) \geq (1 - 1.01\delta) afs(c''_1)$ implying $|c'_1 c''_1| \leq \frac{1.01\delta}{1 - 1.01\delta} afs(c''_1) \leq 2\delta \cdot |c''_1|.$

Another implication of $|s_x c''_1| \leq |s_x z| = 4\sqrt{\delta} \cdot afs(s_x)$ is $afs(c''_1) \geq \left(1 - 4\sqrt{\delta}\right) \cdot afs(s_x)$. Thus, we have $|c_1 c''_1| \leq \tan(\sqrt{\delta}) \cdot |s_x c''_1| \leq 1.1 \cdot \sqrt{\delta} \cdot 4 \cdot \sqrt{\delta} \cdot afs(s_x) \leq \frac{4.4 \delta}{1 - 4\sqrt{\delta}} \cdot afs(c''_1) \leq 5\delta \cdot afs(c''_1)$ which is upper-bounded by $\frac{5\delta}{1 - 10\delta} \cdot afs(c''_1) \leq 6 \cdot \delta \cdot afs(c''_1)$.

Finally, the triangle inequality implies $|c_1 c'_1| \leq |c_1 c''_1| + |c''_1 c'_1| \leq 8 \cdot \delta \cdot afs(c'_1)$.

This is a contradiction to Lemma 24 because $c'_1 \in \Lambda$ and $c_1 \in c$.

On the other hand, to show that $c \cap C_{s_x} \cap s_x s_y = \emptyset$, we assume for the sake of contradiction that there is a point $c_2$ inside the cube $c$ such that $c_2 \in C_{s_x} \cap s_x s_y$. The approach is as follows: We consider a path $\gamma_{c_2 u} \subset C$ (see below for the details of how to construct $\gamma_{c_2 u}$) between $c_2$ and a point $u \in xy$. As $c_2 \in c$ and $u \in \Lambda$, there is a point $v \in \gamma_{c_2 u} \cap \Gamma$, see Figure 8. Lemma 4 implies that there is a sample point $s_v$ such that $|s_v u| \leq 1.17 \cdot \delta \cdot afs(s_v)$. The construction of $\gamma_{c_2 u}$ ensures that $s_v$ lies in the interior of $C_{s_x}$ and thus in the interior of $c$. This is a contradiction to the assumption that there is no $s'_y \in C \cap A(s_x) \setminus \{s_y\}$ concluding the proof.

The construction of $\gamma_{c_2 u}$ is as follows: Let $\ell_{c_2}$ be the line that subtends an angle of $45^\circ$ with $xy$ such that $c_2 \in \ell_{c_2}$. If $\ell_{c_2}$ intersects $xy$ in a point $u$ we define $\gamma_{c_2 u} = c_2 u$. 
Otherwise, let $\gamma_{c_{2u}} := c_{2y}$. Lemma 24 implies $|c_{2u}| \geq |c_{2v}| \geq 6 \cdot \delta \cdot afs(v)$. Thus, the ball with radius $1.17 \cdot \delta \cdot afs(s_v)$ and centered at $v$ lies in the interior of $C_{s_x}$. This concludes the proof.

By construction, there is a local edge $(s_1, s_2) \in E_{loc}$ for any two sample points $s_1, s_2 \in S^{sub}$ with $|s_1, s_2| \leq \frac{1}{3} \cdot \delta \cdot \min\{afs(s_1), afs(s_2)\}$. Combining this with Lemma 26, we now give the proof of Lemma 23 and thus complete the analysis of the approximation quality of our algorithm.

**Lemma 23:** For all $s_1, s_2 \in S^{sub}$, $L(s_1, s_2) \leq (1 + O(\sqrt{\varepsilon})) \cdot L^*_T(s_1, s_2)$.

**Proof.** The approach is the following: Let $\lambda$ be a geodesic shortest path between $s_1$ and $s_2$. By applying Lemma 8 (BE1), we show how to select another sample point $s \in S^{sub}$ and a path $\lambda_{s_1,s} \subset C^*$ between $s_1$ and $s$ such that $|\lambda_{s_1,s}| \leq (1 + O(\sqrt{\varepsilon})) \cdot (L^*_T(s_1, s_2) - L^*_T(s, s_2))$. We then repeat this construction for $s_1 \leftarrow s$ until $s = s_2$. Inductively, we thus construct a path between $s_1$ and $s_2$ that is no longer than $(1 + O(\sqrt{\varepsilon})) \cdot |\lambda|$. To determine the sample point $s$ described above, we first fix $x \in \lambda$ such that $L^*(s_1, x) = \frac{1}{2} \cdot \sqrt{\delta} \cdot \min\{afs(s_1), afs(x)\}$ (if no such point exists in the initial iteration, $s_1$ and $s_2$ are close enough to induce a local edge and the lemma follows). We then distinguish whether or not $x \in \Gamma$ holds.

**Case 1 ($x \in \Gamma$):** Lemma 4 implies that there is a closest sample point $s_x \in S^{sub}$ to $x$ with $|xs_x| \leq 1.17 \cdot \delta \cdot afs(s_x)$. This implies $|s_1s_x| \leq \frac{1}{3} \cdot \sqrt{\delta} \cdot \min\{afs(s_1), afs(s_x)\}$, i.e., $(s_1, s_x) \in E_{loc}$. The length of $(s_1, s_x)$ is upper-bounded by $(1 + O(\sqrt{\varepsilon})) \cdot (L^*_T(s_1, s_x) - L^*_T(s_1, s_x))$.

Furthermore, $afs(s_x) \leq (1 + 1.17 \cdot \delta) \cdot afs(x)$ and $afs(x) \leq (1 - \frac{1}{6} \cdot \sqrt{\delta}) \cdot afs(s_1)$. This implies that $L^*_T(s_1, s_x) - L^*_T(s_1, s_x)$ is lower-bounded by $(\frac{1}{6} \cdot \sqrt{\delta} - 1.17 \cdot (1 + O(\sqrt{\varepsilon})) \cdot (1 + 1.17 \cdot \delta)) \cdot \min\{afs(s_1), afs(x)\} \geq \frac{1}{6} \cdot \sqrt{\delta} - 1.17 \cdot (1 + O(\sqrt{\varepsilon})) \cdot \min\{afs(s_1), afs(x)\}$. A similar argument implies $|s_1s_x| \leq \frac{1}{6} \cdot \sqrt{\delta} \cdot (1 + O(\sqrt{\varepsilon})) \cdot \min\{afs(s_1, x), afs(x)\}$.

Hence, $L^*_T(s_1, s_x) - L^*_T(s_1, s_x) \geq (1 - O(\sqrt{\varepsilon})) \cdot \min\{afs(s_1, x), afs(x)\}$ or, equivalently, $|s_1s_x| \leq (1 + O(\sqrt{\varepsilon})) \cdot (L^*_T(s_1, s_x) - L^*_T(s_1, s_x))$ which is upper-bounded by $(1 + O(\sqrt{\varepsilon})) \cdot |s_1s_x|$.

**Case 2 ($x \notin \Gamma$):** The main idea in this case is to extend the path to be approximated by “jumping” over $x$, i.e., to find two points $y$ and $z$ on $\Gamma$ that immediately precede resp. follow $x$ while traversing $\lambda$. These points then induce sample points $s_y$ and $s_z$ in $S^{sub}$; if $(s_y, s_z) \in E_{loc} \cup E_{brs}$, we use such an edge, otherwise we determine a point $s'_z$ in the approximate visibility neighborhood of $s_y$ and use this point instead of $s_z$; see Case 2.2.2. More precisely, let $y \in \lambda \cap \Gamma$ be last point on $\Gamma$ encountered while traversing $\lambda$ toward $x$, see Figure 9. Lemma 4 implies that there is a sample point $s_y \in S^{sub}$ such that $|ys_y| \leq 1.17 \cdot \delta \cdot afs(s_y)$. We distinguish whether or not $L^*_T(s_1, y) \geq \frac{1}{12} \cdot \sqrt{\delta} \cdot \min\{afs(s_1), afs(y)\}$ holds.
Figure 9: Construction of $\lambda_{s_1 s'_z}$ in the configuration of Case 2.2.2.

**Case 2.1** ($L^*_1(s_1, y) \geq \frac{1}{12} \cdot \sqrt{\delta} \cdot \min\{afs(s_1), afs(y)\}$): As the sample point $s_y$ was chosen such that $|y s_y| \leq 1.17 \cdot \delta \cdot afs(s_y)$ holds, we can argue as in Case 1 to show $|s_1 s_y| \leq \frac{1}{6} \cdot \sqrt{\delta} \cdot \min\{afs(s_1), afs(s_y)\}$ holds as well. Hence, our algorithms constructs a local edge between $s_1$ and $s_y$, i.e., we have $(s_1, s_y) \in E_{loc}$. Proceeding along the same lines as in Case 1, we derive $(1 + O(\sqrt{\delta}))(L^*_1(s_1, s_2) - L^*_1(s_1, s_y)).$

**Case 2.2** ($L^*_1(s_1, y) < \frac{1}{12} \cdot \sqrt{\delta} \cdot \min\{afs(s_1), afs(y)\}$): Let $z \in \lambda \cap \Gamma$ be the first point on $\Gamma$ encountered after leaving $x$. As $z$ lies on $\Gamma$, Lemma 4 implies that there is a sample point $z \in S^{ub}$ with $|z s_z| \leq 1.17 \cdot \delta \cdot afs(s_z)$.

We now show that there is a local edge connecting $s_1$ and $s_y$. For this, we note that the triangle inequality yields that $|s_1 s_y|$ is upper-bounded by $|s_1 y| + |y s_y| \leq \frac{1}{12} \cdot \sqrt{\delta} \cdot \min\{afs(s_1), afs(y)\} + 1.17 \cdot \delta \cdot afs(s_y)$. Using the 1-Lipschitz continuity of $afs(\cdot)$ we can upper-bound this term by $\frac{1}{12} \cdot \sqrt{\delta} \cdot \min\{afs(s_1), afs(s_y)\}$. Thus, $(s_1, s_y) \in E_{loc}$.

To complete bounding the distance between $s_1$ and $s_z$, we distinguish whether or not $|s_y s_z| \leq \frac{1}{6} \sqrt{\delta} \min\{afs(s_y), afs(s_z)\}$ holds:

**Case 2.2.1** ($|s_y s_z| \leq \frac{1}{6} \cdot \sqrt{\delta} \cdot \min\{afs(s_y), afs(s_z)\}$): As $s_y$ and $s_z$ are assumed to be close enough, we know that our algorithm constructs a local edge between these points, i.e., we have $(s_y, s_z) \in E_{loc}$. Using the 1-Lipschitz continuity of $afs(\cdot)$, we obtain $|y s_y|, |z s_z| \in O(\sqrt{\delta}) \cdot |y z|$. Using the same approach as in Case 1, we can derive that $|s_1 s_y| + |s_y s_z| \leq (1 + O(\sqrt{\delta})) \cdot (L^*_1(s_1, s_2) - L^*_1(s_1, s_y))$ holds.

**Case 2.2.2** ($|s_y s_z| > \frac{1}{6} \cdot \sqrt{\delta} \cdot \min\{afs(s_y), afs(s_z)\}$): As $x$ and $y$ can see each other, Lemma 26 implies that there is a sample point $s'_z \in A(s_y)$ such that $\angle(s'_z, s_y, s_z) \leq O(\sqrt{\delta})$ at which $s_z$ is the closest sample point to $z$. This implies that Properties (B) and (D) are fulfilled. By applying similar approaches as we applied in the proof of Lemma 16 (see Figure 4) we construct $z'$ such that Properties (A) and (C) are fulfilled.

**Property (A):** $\angle(s'_z, s_y, z') \in O(\sqrt{\delta})$,

**Property (B):** $|s_y s'_z| \geq \Omega(\sqrt{\delta}) \min\{afs(s_y), afs(s'_z)\}$,

**Property (C):** $L^*_1(s'_z, z') \leq O(\sqrt{\delta}) \cdot |s_y s'_z|$, and
Property (D): \((s_y, s_z') \in E_{\text{loc}} \cup E_{\text{bri}}\).

Based on Properties (A)–(D), we then construct the desired path \(\bar{x}_{s_1, s_z'} \subset G^*\) such that \(|\bar{x}_{s_1, s_z'}| \leq (1 + O(\sqrt{\varepsilon})) \cdot (L^*_T(s_1, s_2) - L^*_T(s, s_2))\) holds as follows:

By Lipschitz continuity of \(afs(\cdot, \cdot)\), we obtain that Property (B) implies \(\Omega(\sqrt{\varepsilon}) \cdot afs(s_y) \leq |s_y s'_z|\) which is equivalent to \(afs(s_y) \leq \frac{|s_y s'_z|}{\Omega(\sqrt{\varepsilon})}\). By combining
\[
|ys_y| \leq 1.17 \cdot \delta \cdot afs(s_y)\]
with \(afs(s_y) \leq \frac{|s_y s'_z|}{\Omega(\sqrt{\varepsilon})}\), we obtain that
\[
|ys_y| \leq \frac{1.17 \cdot \delta}{\Omega(\sqrt{\varepsilon})} \cdot |s_y s'_z|\]
holds.

Furthermore, Property (C) implies \(|s_z' z'| \leq \Omega(\sqrt{\varepsilon}) \cdot |s_y s'_z|\) because \(|s_y s'_z| \leq L^*_T(s, s'_z)\). The triangle inequality implies \(|s_y s'_z| \leq |s_y y| + |y z'| + |z' s'_z|\) which is equivalent to \(|s_y s'_z| - |s_y y| - |z' s'_z| \leq |y z'|\). Thus, we can upper-bound
\[
(1 - O(\sqrt{\varepsilon})) \cdot |s_y s'_z| \leq |y z'|\]
and we can lower-bound
\[
|s_y z'| \cdot (1 - O(\sqrt{\varepsilon})) \cdot |s_y s'_z| \geq \left(1 - O(\sqrt{\varepsilon})\right) \cdot |y z'|\]
which implies \(|y s_y| + |y z'| \geq \left(1 - O(\sqrt{\varepsilon})\right) \cdot |y z'|\). Furthermore, based on \(|y s_y| \in O(\sqrt{\varepsilon}) \cdot |y z'|\), we can show \(|y z'| \geq \left(1 - O(\sqrt{\varepsilon})\right) \cdot |s_y z'|\)
which implies \(|y s_y| + |y z'| \geq \left(1 - O(\sqrt{\varepsilon})\right) \cdot |s_y z'|\)
and we can lower-bound \(L^*_T(s_1, s_2) - L^*_T(s'_z, s_2) \geq |s_1 s_y| + (1 - O(\sqrt{\varepsilon})) \cdot |s_y z'| - L^*_T(s'_z, s'_z)\).

Furthermore, Property (A), i.e., \(\angle(s'_z, s_y, z') \leq O(\sqrt{\varepsilon})\), implies \(|s_y z'| \geq \left(1 - O(\sqrt{\varepsilon})\right) \cdot |s_y s'_z|\)
and we can lower-bound \(L^*_T(s_1, s_2) - L^*_T(s'_z, s_2) \geq |s_1 s_y| + (1 - O(\sqrt{\varepsilon})) \cdot |s_y s'_z| - L^*_T(z', s'_z)\).

Finally, Property (B), i.e., \(L^*_T(s'_z, z') \leq O(\delta) \cdot |s_y s'_z|\), implies \(L^*_T(s_1, s_2) - L^*_T(s'_z, s_2) \geq |s_1 s_y| + (1 - O(\sqrt{\varepsilon})) \cdot |s_y s'_z| - O(\sqrt{\varepsilon}) |s_y s'_z| \geq (1 - O(\sqrt{\varepsilon})) |s_1 s_y| + |s_y s'_z|\).

Hence, we have \(\left(1 - O(\sqrt{\varepsilon})\right)^{-1} \cdot (L^*_T(s_1, s_2) - L^*_T(s'_z, s_2) \geq |s_1 s_y| + |s_y s'_z|\) which is equivalent to \((1 + O(\sqrt{\varepsilon})) \cdot (L^*_T(s_1, s_2) - L^*_T(s'_z, s_2)) \geq |s_1 s_y| + |s_y s'_z|\).

Thus, Lemma 3 (LE2) implies \(\mathcal{L}(s_1, s_2) \leq (1 + O(\sqrt{\varepsilon})) \cdot (1 + O(\sqrt{\varepsilon})) \cdot L^*_T(s_1, s_2) \leq (1 + O(\sqrt{\varepsilon})) \cdot L^*_T(s_1, s_2).

**Corollary 5.** For all \(s_1, s_2 \in S\), \(\mathcal{L}(s_1, s_2) \leq (1 + O(\sqrt{\varepsilon})) \cdot L^*_T(s_1, s_2)\).
In conclusion, the discussion in this section constitutes a proof of Lemma 10, i.e., we have shown that \( L(\cdot, \cdot) \) is a \((1 \pm O(\sqrt{\varepsilon}))\)-approximation of \( L^*_1(\cdot, \cdot) \). Together with Lemma 9, where we showed the running time of our algorithm to be in \( O(n^{5/2} \log^2 n) \), this constitutes a proof of our main result:

**Theorem 1** There is a global and shape-independent constant \( \varepsilon_0 > 0 \) such that it holds for \( \varepsilon \leq \varepsilon_0 \): Given an \( \varepsilon \)-sample \( S \) of a set of smooth obstacles in \( \mathbb{R}^3 \), we can compute \((1 \pm O(\sqrt{\varepsilon}))\)-approximations of all \( \binom{n}{2} \) distances in \( O(n^{5/2} \log^2 n) \) time, where \( n := |S| \).

**Acknowledgement** We thank an anonymous reviewer for his/her detailed comments that helped improving the presentation of this paper.

**References**


