NEW LOWER BOUNDS FOR THE NUMBER OF PSEUDOLINE ARRANGEMENTS

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Abstract. Arrangements of lines and pseudolines are fundamental objects in discrete and computational geometry. They also appear in other areas of computer science, for instance in the study of sorting networks. Let \( B_n \) be the number of nonisomorphic arrangements of \( n \) pseudolines and let \( b_n = \log_2 B_n \). The problem of estimating \( B_n \) was posed by Knuth in 1992. Knuth conjectured that \( b_n \leq \left( \frac{n}{2} \right) + o(n^2) \) and also derived the first upper and lower bounds: \( b_n \leq 0.7924(n^2 + n) \) and \( b_n \geq n^2/6 - O(n) \). The upper bound underwent several improvements, \( b_n \leq 0.6974 n^2 \) (Felsner, 1997), and \( b_n \leq 0.6571 n^2 \) (Felsner and Valtr, 2011), for large \( n \). Here we show that \( b_n \geq cn^2 - O(n \log n) \) for some constant \( c > 0 \). In particular, \( b_n \geq 0.2083 n^2 \) for large \( n \). This improves the previous best lower bound, \( b_n \geq 0.1887 n^2 \), due to Felsner and Valtr (2011). Our arguments are elementary and geometric in nature. Further, our constructions are likely to spur new developments and improved lower bounds for related problems, such as in topological graph drawings.

Keywords: counting, pseudoline arrangement, recursive construction.

1 Introduction

Arrangements of pseudolines. A pseudoline in the Euclidean plane is an \( x \)-monotone curve extending from negative infinity to positive infinity. An (Euclidean) arrangement of pseudolines is a family of pseudolines where each pair of pseudolines has a unique point of intersection (called ‘vertex’). An arrangement is simple if no three pseudolines have a common point of intersection, see Fig. 1 (left). Throughout this paper the term arrangement always means simple arrangement if not specified otherwise.

Figure 1: Left: A simple arrangement \( \mathcal{A} \). Center: Wiring diagram of \( \mathcal{A} \). Right: An arrangement \( \mathcal{A}' \) that is not isomorphic to the arrangement \( \mathcal{A} \) on the left.

There are several combinatorial representations (and encodings) of pseudoline arrangements. These representations help one count the number of arrangements. Three clas-
sic representations are allowable sequences (introduced by Goodman and Pollack [10, 11]), wiring diagrams [8], and zonotopal tilings [7].

A simple allowable sequence is a sequence $\Sigma$ of $\binom{n}{2} + 1$ permutations of \{1, 2, \ldots, n\} satisfying two properties: (i) The first element of $\Sigma$ is the identity permutation $(1, 2, \ldots, n)$ and the last element of $\Sigma$ is the reverse permutation $(n, \ldots, 2, 1)$; and (ii) Two consecutive permutations in $\Sigma$ differ by the reversal of an adjacent pair $ij$, where $i < j$ [6]. A wiring diagram is a Euclidean arrangement of pseudolines consisting of piece-wise linear ‘wires’, each horizontal except for a short segment where it crosses another wire. Each pair of wires cross exactly once; see Fig. 1 (center). Wiring diagrams are also known as reflection networks, i.e., networks that bring $n$ wires labeled from 1 to $n$ into their reflection by means of performing switches of adjacent wires; see [14, p. 35]. Lastly, they are also known under the name of primitive sorting networks; see [15, Ch. 5.3.4]. The number of (simple) allowable sequences is denoted by $A_n$ (sequence A005118 in [21]). Stanley [22] established the following closed formula for $A_n$ (see also Table 1):

$$A_n = \frac{\binom{n}{2}!}{\prod_{k=1}^{n-1} (2n - 2k - 1)^k}.$$ 

Two arrangements are isomorphic, i.e., considered the same, if they can be mapped onto each other by a homeomorphism of the plane [9]; see Figures 1 and 2. Equivalently, two arrangements are isomorphic if there is an isomorphism between the induced cell decomposition [7, Ch. 6]. The number of nonisomorphic arrangements of $n$ pseudolines is denoted by $B_n$ (sequence A006245 in [21]); this is the number of equivalence classes of all arrangements of $n$ pseudolines; see [14, p. 35]. It is worth pointing out that for $A_n$, the left to right order of the vertices in the arrangement plays a role while for $B_n$ only the order of vertices along each particular pseudoline is important, i.e., the relative position of two vertices from distinct pairs of pseudolines does not matter. Many allowable sequences may correspond to the same arrangement. See Fig. 2 for an illustration of this concept and the correspondence with allowable sequences.

Figure 2: $A_1$, $A_2$ and $A_3$ are three arrangements with four pseudolines. $A_1$ and $A_2$ are isomorphic since the positions of the vertices 23 (in red) and 14 (in blue) can be switched. $A_3$ is nonisomorphic to $A_2$ (and $A_1$) since the positions of the vertices 23 (in red) and 34 (in green) cannot be switched because they have a common pseudoline. The corresponding allowable sequences are:

$A_1$: 1234 $\rightarrow$ 2134 $\rightarrow$ 2314 $\rightarrow$ 3214 $\rightarrow$ 1324 $\rightarrow$ 3421 $\rightarrow$ 3421 $\rightarrow$ 4321.

$A_2$: 1234 $\rightarrow$ 1234 $\rightarrow$ 1324 $\rightarrow$ 2314 $\rightarrow$ 3214 $\rightarrow$ 3421 $\rightarrow$ 3421 $\rightarrow$ 4321.

$A_3$: 1234 $\rightarrow$ 1234 $\rightarrow$ 1324 $\rightarrow$ 2314 $\rightarrow$ 3214 $\rightarrow$ 3421 $\rightarrow$ 3421 $\rightarrow$ 4321.
Here we study the growth rate of \( B_n \); so let\(^1 \) \( b_n = \log B_n \). Knuth [14] conjectured that 
\( b_n \leq \binom{n}{2} + o(n^2) \); see also [8, p. 147] and [6, p. 259]. This conjecture is still open.

**Upper bounds on the number of pseudoline arrangements.** Felsner [6] used a horizontalencoding of an arrangement in order to estimate \( B_n \). An arrangement can be represented by a sequence of horizontal cuts. The \( i \)th cut is the list of pseudolines crossing the \( i \)th pseudoline in the order of the crossings. Using this approach, Felsner [6, Thm. 1] obtained the upper bound 
\[ b_n \leq 0.7213(n^2 - n) \]
he further refined this bound by using *replace matrices*. A replace matrix is a binary \( n \times n \) matrix \( M \) with the properties \( \sum_{j=1}^{n} m_{ij} = n - i \) for all \( i \) and \( m_{ij} \geq m_{ji} \) for all \( i < j \). Using this technique, the author established the upper bound 
\[ b_n \leq 0.6974n^2 \] [6, Thm. 2].

In his seminal paper on the topic, Knuth [14] took a vertical approach for encoding arrangements. Let \( \mathcal{A} \) be an arrangement of \( n \) pseudolines \( \{\ell_1, \ldots, \ell_n\} \). By adding pseudoline \( \ell_{n+1} \) to \( \mathcal{A} \), we get \( \mathcal{A}' \), an arrangement of \( n + 1 \) pseudolines. The course of \( \ell_{n+1} \) describes a vertical cutpath from top to bottom. The number of cutpaths of \( \mathcal{A} \) is exactly the number of arrangements \( \mathcal{A}' \) such that \( \mathcal{A}' \setminus \{\ell_{n+1}\} \) is isomorphic to \( \mathcal{A} \). Let \( \gamma_n \) denote the maximum number of cutpaths in an arrangement of \( n \) pseudolines. Therefore, one has 
\[ B_{n+1} = \gamma_n \cdot B_n \]
and \( B_3 = 2 \). Knuth [14] proved that 
\[ \gamma_n \leq 3^n \]
concluding that \( B_n \leq 3^{(n+1)/2} \) and thus 
\[ b_n \leq 0.5(n^2 + n) \log 3 \leq 0.7924(n^2 + n) \]; this computation can be streamlined so that it yields 
\[ b_n \leq 0.7924n^2 \] see [9]. Knuth also conjectured that 
\[ \gamma_n \leq n \cdot 2^n \]
but this was refuted by Ondřej Bílka in 2010 [9]; see also [8, p. 147]. The current best estimates on \( \gamma_n \) are 
\[ 2.076^n \leq \gamma_n \leq 4n \cdot 2.487^n \]
see [9]. The latter inequality yields 
\[ b_n \leq 0.6571n^2 \]
which is the current best upper bound.

**Lower bounds on the number of pseudoline arrangements.** Knuth [14, p. 37] gave a recursive construction in the setting of reflection networks. The number of nonisomorphic arrangements of \( n \) pseudolines in his construction, \( T(n) \), obeys the recurrence
\[ T(n) \geq 2^{n^2/8 - n/4} \cdot T(n/2). \]
By induction this yields 
\[ T(n) \geq 2^{n^2/6 - 5n/2}, \] therefore \( B_n \geq 2^{n^2/6 - 5n/2} \).

Matoušek sketched another recursive construction [18, Sec. 6.2], see Fig. 3. Let \( n \) be a multiple of 3 and \( m = \frac{n}{3} \) (assume that \( m \) is odd). The \( 2m \) lines in the two extreme bundles form a regular grid of \( m^2 \) points. The lines in the central bundle are incident to 
\[ \frac{3m^2 + 1}{4} \]
of these grid points. At each such point, there are 2 choices; going below it or above it, thus creating at least 
\[ \frac{3m^2}{4} = \frac{3(n/3)^2}{4} = \frac{n^2}{12} \]
binary choices. Thus \( T(n) \) obeys the recurrence
\[ T(n) \geq 2^{n^2/12} \cdot (T(n/3))^3, \]
which by induction yields 
\[ T(n) \geq 2^{n^2/8}, \] implying \( B_n \geq 2^{n^2/8} \).

Felsner and Valtr [9] used rhombic tilings of a centrally symmetric hexagon in an elegant recursive construction for a lower bound on \( B_n \). Consider a set of \( i + j + k \) pseudolines

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\(^1\)Throughout this paper, \( \log x \) and \( \ln x \) are the logarithms of \( x \) in base 2 and \( e \), respectively.
partitioned into the following three parts: \(\{1, \ldots, i\}, \{i+1, \ldots, i+j\}, \{i+j+1, \ldots, i+j+k\}\), see Fig. 4. A partial arrangement is called *consistent* if any two pseudolines from two different parts always cross but any two pseudolines from the same part never cross.

![Figure 3: Grid construction for a lower bound on \(B_n\).](image)

![Figure 4: The hexagon \(H(5, 5, 5)\) with one of its rhombic tilings and a consistent partial arrangement corresponding to the tiling. This figure is reproduced from [9].](image)

The zonotopal duals of consistent partial arrangements are rhombic tilings of the centrally symmetric hexagon \(H(i, j, k)\) with side lengths \(i, j, k\). The enumeration of rhombic tilings of \(H(i, j, k)\) was solved by MacMahon [17] (see also [3]), who proved that the number of tilings is

\[
P(i, j, k) = \prod_{a=0}^{i-1} \prod_{b=0}^{j-1} \prod_{c=0}^{k-1} \frac{a + b + c + 2}{a + b + c + 1}.
\]

An approximation using integral calculus [9] shows that

\[
\ln P(n, n, n) \approx \left(\frac{9}{2} \ln 3 - 6 \ln 2\right) n^2.
\]

Assuming \(n\) to be a multiple of 3 in the recursion step, the construction yields the recurrence

\[
T(n) \geq P\left(\frac{n}{3}, \frac{n}{3}, \frac{n}{3}\right) \cdot \left(T\left(\frac{n}{3}\right)\right)^3.
\]

By induction, formula (2) together with the recurrence (3) yield the lower bound \(b_n \geq 0.1887 n^2\) for large \(n\); this is the previous best lower bound.
Table 1 shows the exact values of $A_n$ and $B_n$, and their growth rate (up to four digits after the decimal point) with respect to $n$, for small values of $n$. The values of $B_n$ for $n = 1$ to 9 are from [14, p. 35] and the values of $B_{10}$, $B_{11}$, and $B_{12}$ are from [6, 23], and [21], respectively; the values of $B_{13}$, $B_{14}$, and $B_{15}$ have been added recently, see [13, 21]. Observe that $A_n$ grows much faster than $B_n$.

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<th>$B_n$</th>
<th>$\log \frac{B_n}{n^2}$</th>
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Table 1: Values of $A_n$ and $B_n$ for small $n$.

Our results. Here we extend the method used by Matoušek in his grid construction; observe that it uses lines of 3 slopes. In Sections 2 (the 2nd part) and 3, we use lines of 6 and 12 different slopes in hexagonal type constructions; yielding lower bounds $b_n \geq 0.1981 n^2$ and $b_n \geq 0.2083 n^2$ for large $n$, respectively. In Sections A and B of the Appendix, we use lines of 8 and 12 different slopes in rectangular type constructions; yielding the lower bounds $b_n \geq 0.1999 n^2$ and $b_n \geq 0.2053 n^2$ for large $n$, respectively. While the construction in Section 3 gives a better bound, the one in Section 2 is easier to analyze; (the results in Section 2, A and B have appeared in [3]). For each of the two styles, rectangular and hexagonal, the constructions are presented in increasing order of complexity. Our main result is summarized in the following.

**Theorem 1.** Let $B_n$ be the number of nonisomorphic arrangements of $n$ pseudolines. Then $B_n \geq 2^{cn^2 - O(n \log n)}$, for some constant $c > 0.2083$. In particular, $B_n \geq 2^{0.2083 n^2}$ for large $n$.

Outline of the proof. We construct a line arrangement using lines of $k$ different slopes (for a small $k$). The final construction will be obtained by a small clockwise rotation, so that there are no vertical lines. Let $m = \lfloor n/k \rfloor$ or $m = \lceil n/k \rceil - 1$ (whichever is odd). Each bundle consists of $m$ equidistant parallel lines in the corresponding strip; remaining lines are discarded, or not used in the counting. An $i$-wise crossing is an intersection point of exactly $i$ lines. Let $\lambda_i(m)$ denote the number of $i$-wise crossings in the arrangement.
where each bundle consists of $m$ lines. Our goal is to estimate $\lambda_i(m)$ for each $i$. Then we can locally replace the lines around each $i$-wise crossing with any of the $B_i$ nonisomorphic pseudoline arrangements; and further apply recursively this construction to each of the $k$ bundles of parallel lines exiting this junction. This yields a simple pseudoline arrangement for each possible replacement choice. Consequently, the number of nonisomorphic pseudoline arrangements in this construction, denoted by $T(n)$, satisfies the recurrence:

$$T(n) \geq F(n) \left[ T\left(\frac{n}{k}\right) \right]^k,$$

(4)

where $F(n)$ is a multiplicative factor counting the number of choices in this junction:

$$F(n) \geq \prod_{i=3}^{k} B_i^\lambda_i(n).$$

(5)

**Related work.** In a comprehensive recent paper, Kynčl [16] obtained estimates on the number of isomorphism classes of simple topological graphs that realize various graphs. The author remarks that it is probably hard to obtain tight estimates on this quantity, “given that even for pseudoline arrangements, the best known lower and upper bounds on their number differ significantly”. While our improvements aren’t spectacular, it seems however likely that some of the techniques we used here can be employed to obtain improved lower bounds for topological graph drawings too.

**Notations and formulas used.** For a figure $F$, let $\text{per}(F)$ denote its perimeter, i.e., the length of its boundary. For two similar polygonal figures $F, F'$, let $\rho(F, F')$ denote their similarity ratio, i.e., the ratio between the lengths of corresponding sides of $F$ and $F'$ (which is equal to $\text{per}(F)/\text{per}(F')$). For a planar region $R$, let $\text{area}(R)$ denote its area. By slightly abusing notation, let $\text{area}(i, j, k)$ denote the area of the triangle made by three lines $\ell_i, \ell_j$ and $\ell_k$. Assume that the equations of the three lines are $\alpha_s x + \beta_s y + \gamma_s = 0$, for $s = 1, 2, 3$, respectively. Then the respective triangle area can be computed as follows (for instance, see [19] or [20, pp. 27–28]:

$$\text{area}(i, j, k) = \frac{A^2}{2|C_1C_2C_3|},$$

where

$$A = \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix},$$

$$C_1 = (\alpha_2 \beta_3 - \beta_2 \alpha_3),$$

$$C_2 = -(\alpha_1 \beta_3 - \beta_1 \alpha_3),$$

$$C_3 = (\alpha_1 \beta_2 - \beta_1 \alpha_2).$$

Let $P(i, j, g, h)$ denote the parallelogram made by the pairs of parallel lines $\ell_i \parallel \ell_j$ and $\ell_g \parallel \ell_h$. A *strip* is the set of points in between two parallel lines.
Preliminary constructions

Warm-up: a rectangular construction with 4 slopes. We start with a simple rectangular construction with 4 bundles of parallel lines whose slopes are 0, ∞, ±1; see Fig. 5. Let $U = [0, 1]^2$ be the unit square we work with. The axes of all four strips are incident to the center of $U$.

For $i = 3, 4$, let $a_i$ denote the area of the region covered by exactly $i$ of the 4 strips. It is easy to see that $a_3 = a_4 = 1/2$, and obviously $a_3 + a_4 = \text{area}(U) = 1$. Observe that $\lambda_i(m)$ is proportional to $a_i$, for $i = 3, 4$; taking the boundary effect into account, we have

$$\lambda_3(m) = a_3 m^2 - O(m) = \frac{m^2}{2} - O(m), \quad \lambda_4(m) = a_4 m^2 - O(m) = \frac{m^2}{2} - O(m).$$

Since $m = n/4$, $\lambda_i$ can be also viewed as a function of $n$. Therefore

$$\lambda_3(n) = \frac{n^2}{32} - O(n), \quad \lambda_4(n) = \frac{n^2}{32} - O(n),$$

and so the multiplicative factor in Eq. (4) is bounded from below as follows:

$$F(n) \geq \prod_{i=3}^{4} B_i^{\lambda_i(n)} \geq 2^{\frac{n^2}{32} - O(n)} \cdot 8^{\frac{n^2}{32} - O(n)} = 2^{\frac{n^2}{8} - O(n)}. \quad (6)$$

Applying (4) for $k = 4$ yields

$$T(n) \geq F(n) \cdot (T(n/4))^4 \geq 2^{\frac{n^2}{8} - O(n)} \cdot (T(n/4))^4.$$

By induction on $n$, the resulting lower bound is $T(n) \geq 2^{n^2/6 - O(n \log n)}$; this matches the constant 1/6 in Knuth’s lower bound described in Section 1.
Hexagonal construction with 6 slopes. We next describe and analyze a hexagonal construction with lines of 6 slopes, namely 6 bundles of parallel lines whose slopes are 0, ∞, ±1/√3, ±√3. Let $H$ be a regular hexagon whose side has unit length. The axes of the 6 strips containing the bundles of lines are incident to the center of $H$; see Fig. 6 (left). This construction yields the lower bound $b_n \geq 0.1981 n^2$ for large $n$.

Let $\mathcal{L} = \bigcup_{i=1}^{6} \mathcal{L}_i$ be the partition of the lines into six bundles of parallel lines. The $m$ lines in $\mathcal{L}_i$ are contained in the strip bounded by the two lines $\ell_{2i-1}$ and $\ell_{2i}$, for $i = 1, \ldots, 6$. We refer to lines in $\mathcal{L}_1 \cup \mathcal{L}_3 \cup \mathcal{L}_5$ as the primary lines, and to lines in $\mathcal{L}_2 \cup \mathcal{L}_4 \cup \mathcal{L}_6$ as secondary lines. Three strips are bounded by the pairs of lines supporting opposite sides of $H$, while the other three strips are bounded by the pairs of lines supporting opposite short diagonals of $H$.

Assume a coordinate system where the lower left corner of $H$ is at the origin, and the lower side of $H$ lies along the $x$-axis. The equation of line $\ell_i$ is $\alpha_i x + \beta_i y + \gamma_i = 0$, with $\alpha_i, \beta_i, \gamma_i$, for $i = 1, \ldots, 12$, given in Fig. 6 (right).

Note that the distance between consecutive lines in any of the bundles of

- primary lines is $\frac{\sqrt{3}}{m} \left( 1 - O \left( \frac{1}{m} \right) \right)$;
- secondary lines is $\frac{1}{m} \left( 1 - O \left( \frac{1}{m} \right) \right)$.

Let $\sigma_0 = \sigma_0(m)$ and $\delta_0 = \delta_0(m)$ denote the basic parallelogram and triangle, respectively, determined by consecutive lines in $\mathcal{L}_1 \cup \mathcal{L}_3 \cup \mathcal{L}_5$ (in all three possible orientations). The side length of $\sigma_0$ and $\delta_0$ is $\frac{2}{m} \left( 1 - O \left( \frac{1}{m} \right) \right)$. Let $H'$ be the smaller regular hexagon bounded by the short diagonals of $H$; the similarity ratio $\rho(H', H)$ is equal to $\frac{1}{\sqrt{3}}$. $H$ is
the intersection of all three primary strips and \(H'\) is the intersection of all three secondary strips. Recall that (i) the area of an equilateral triangle of side \(s\) is \(\frac{s^2\sqrt{3}}{4}\); and (ii) the area of a regular hexagon of side \(s\) is \(\frac{s^2\sqrt{3}}{2}\); as such, we have

\[
\text{area}(H) = \frac{3\sqrt{3}}{2},
\]

\[
\text{area}(H') = \frac{\text{area}(H)}{3} = \frac{\sqrt{3}}{2},
\]

\[
\text{area}(\delta_0) = \frac{4}{m^2} \frac{\sqrt{3}}{4} \left( 1 - O\left(\frac{1}{m}\right) \right) = \frac{\sqrt{3}}{m^2} \left( 1 - O\left(\frac{1}{m}\right) \right),
\]

\[
\text{area}(\sigma_0) = 2 \cdot \text{area}(\delta_0) = \frac{2\sqrt{3}}{m^2} \left( 1 - O\left(\frac{1}{m}\right) \right).
\]

For \(i = 3, 4, 5, 6\), let \(a_i\) denote the area of the (not necessarily connected) region covered by exactly \(i\) of the 6 strips. The following observations are in order: (i) the six isosceles triangles based on the sides of \(H\) inside \(H\) have unit base and height \(\frac{1}{2\sqrt{3}}\); (ii) the six smaller equilateral triangles incident to the vertices of \(H\) have side-length \(\frac{1}{\sqrt{3}}\). These observations yield

\[
a_3 = \text{area}(H) = \frac{3\sqrt{3}}{2},
\]

\[
a_4 = 6 \cdot \text{area}(3, 5, 7) = 6 \cdot \frac{1}{4\sqrt{3}} = \frac{\sqrt{3}}{2},
\]

\[
a_5 = 6 \cdot \text{area}(3, 7, 11) = 6 \cdot \frac{1}{3\sqrt{3}} = \frac{\sqrt{3}}{2},
\]

\[
a_6 = \text{area}(H') = \frac{\sqrt{3}}{2}.
\]

Observe that \(a_4 + a_5 + a_6 = \text{area}(H)\). Recall that \(\lambda_i(m)\) denote the number of \(i\)-wise crossings where each bundle consists of \(m\) lines. Note that \(\lambda_i(m)\) is proportional to \(a_i\), for \(i = 4, 5, 6\). Indeed, \(\lambda_i(m)\) is equal to the number of \(i\)-wise crossings of lines in \(L_1 \cup L_3 \cup L_5\) that lie in a region covered by \(i\) strips, which is roughly equal to the ratio \(\frac{a_i}{\text{area}(\sigma_0)}\), for \(i = 4, 5, 6\). More precisely, taking also the boundary effect of the relevant regions into account, we obtain

\[
\lambda_4(m) = \frac{a_4}{\text{area}(\sigma_0)} - O(m) = \frac{\sqrt{3}}{2} \frac{m^2}{2\sqrt{3}} - O(m) = \frac{m^2}{4} - O(m),
\]

\[
\lambda_5(m) = \frac{a_5}{\text{area}(\sigma_0)} - O(m) = \frac{m^2}{4} - O(m),
\]

\[
\lambda_6(m) = \frac{a_6}{\text{area}(\sigma_0)} - O(m) = \frac{m^2}{4} - O(m).
\]

For estimating \(\lambda_3(m)\), the situation is little bit different; see Fig. 7. In addition to considering 3-wise crossings of the primary lines (drawn as the crossings of 3 black lines),
we also observe 3-wise crossings of the secondary lines (drawn as the crossings of 3 red lines at the centers of the small equilateral triangles contained in $H'$). It follows that

$$\lambda_3(m) = \frac{a_3}{\text{area}(\sigma_0)} + \frac{\text{area}(H')}{\text{area}(\delta_0)} - O(m) = \frac{3m^2}{4} + \frac{m^2}{2} - O(m) = \frac{5m^2}{4} - O(m).$$

The values of $\lambda_i(m)$, for $i = 3, 4, 5, 6$, are summarized in Table 2; for convenience the linear terms are omitted. Since $m = n/6$, $\lambda_i$ can be also viewed as a function of $n$.

<table>
<thead>
<tr>
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<td>$\lambda_i(m)$</td>
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</tr>
</tbody>
</table>

Table 2: The asymptotic values of $\lambda_i(m)$ and $\lambda_i(n)$ for $i = 3, 4, 5, 6$.

The multiplicative factor in Eq. (4) is bounded from below as follows:

$$F(n) \geq \prod_{i=3}^{6} B_i^{\lambda_i(n)} \geq 2^{5n^2/144} \cdot 8^{n^2/144} \cdot 62^{n^2/144} \cdot 908^{n^2/144} \cdot 2^{-O(n)}. $$

We prove by induction on $n$ that $T(n) \geq 2^{cn^2 - O(n \log n)}$ for a suitable constant $c > 0$.  

Figure 7: Triple incidences of primary lines and triple incidences of secondary lines are drawn in black and red, respectively.
It suffices to choose \( c \) (using the values of \( B_i \) for \( i = 3, 4, 5, 6 \) in Table 1) so that
\[
\frac{8 + \log 62 + \log 908}{144} \geq \frac{5c}{6}.
\]
The above inequality holds if we set \( c = \frac{\log(256 \cdot 62 \cdot 908)}{120} > 0.1981 \), and the lower bound follows.

### 3 Hexagonal construction with 12 slopes

We next describe and analyze a hexagonal construction with lines of 12 slopes, which provides our main result in Theorem 1. Consider 12 bundles of parallel lines whose slopes are \( 0, \infty, \pm \sqrt{3}/5, \pm 1/\sqrt{3}, \pm \sqrt{3}/2, \pm \sqrt{3}, \pm 3\sqrt{3} \). Let \( H \) be a regular hexagon whose side has unit length. The axes of the 12 strips containing the bundles of lines are incident to the center of \( H \); see Figs. 8 and 9. This construction yields the lower bound \( b_n \geq 0.2083 n^2 \) for large \( n \).

\[
\begin{array}{|c|c|c|c|}
\hline
i & \alpha_i & \beta_i & \gamma_i \\
\hline
1 & 3\sqrt{3} & 1 & -\sqrt{3} \\
2 & 3\sqrt{3} & 1 & -3\sqrt{3} \\
3 & \sqrt{3} & 1 & 0 \\
4 & \sqrt{3} & 1 & -2\sqrt{3} \\
5 & \sqrt{3} & 2 & -\sqrt{3} \\
6 & \sqrt{3} & 2 & -2\sqrt{3} \\
7 & 1 & \sqrt{3} & -1 \\
8 & 1 & \sqrt{3} & -3 \\
9 & \sqrt{3} & 5 & -2\sqrt{3} \\
10 & \sqrt{3} & 5 & -4\sqrt{3} \\
11 & 0 & 1 & 0 \\
12 & 0 & 1 & -\sqrt{3} \\
13 & -\sqrt{3} & 5 & -\sqrt{3} \\
14 & -\sqrt{3} & 5 & -3\sqrt{3} \\
15 & -1 & \sqrt{3} & 0 \\
16 & -1 & \sqrt{3} & -2 \\
17 & -\sqrt{3} & 2 & 0 \\
18 & -\sqrt{3} & 2 & -\sqrt{3} \\
19 & -\sqrt{3} & 1 & \sqrt{3} \\
20 & -3\sqrt{3} & 1 & -\sqrt{3} \\
21 & -3\sqrt{3} & 1 & 2\sqrt{3} \\
22 & -3\sqrt{3} & 1 & 0 \\
23 & -1 & 0 & 1 \\
24 & -1 & 0 & 0 \\
\hline
\end{array}
\]

Table 3: Coefficients of the 24 lines.

Assume a coordinate system where the lower left corner of \( H \) is at the origin, and the lower side of \( H \) lies along the \( x \)-axis. Let \( \mathcal{L} = \mathcal{L}_1 \cup \ldots \cup \mathcal{L}_{12} \) be the partition of the lines into twelve bundles of parallel lines. The \( m \) lines in \( \mathcal{L}_i \) are contained in the strip \( \Gamma_i \) bounded by the two lines \( \ell_{2i-1} \) and \( \ell_{2i} \), for \( i = 1, \ldots, 12 \). The equation of line \( \ell_i \) is \( \alpha_i x + \beta_i y + \gamma_i = 0 \), with \( \alpha_i, \beta_i, \gamma_i \), for \( i = 1, \ldots, 24 \), given in Table 3.

\( \Gamma_2, \Gamma_6 \) and \( \Gamma_{10} \) are bounded by the pairs of lines supporting opposite sides of \( H \), while \( \Gamma_4, \Gamma_8 \) and \( \Gamma_{12} \) are bounded by the pairs of lines supporting opposite short diagonals of \( H \). Therefore \( H = \Gamma_2 \cap \Gamma_6 \cap \Gamma_{10} \). We refer to lines in \( \mathcal{L}_2 \cup \mathcal{L}_6 \cup \mathcal{L}_{10} \) as the primary lines, to lines in \( \mathcal{L}_4 \cup \mathcal{L}_8 \cup \mathcal{L}_{12} \) as the secondary lines, and to the rest of the lines as the tertiary lines. Note that the distance between consecutive lines in any of the bundles of

- primary lines is \( \frac{\sqrt{3}}{m} \left( 1 - O \left( \frac{1}{m} \right) \right) \);
- secondary lines is \( \frac{1}{m} \left( 1 - O \left( \frac{1}{m} \right) \right) \);
- tertiary lines is \( \sqrt{\frac{3}{m}} \frac{1}{m} \left( 1 - O \left( \frac{1}{m} \right) \right) \).
We refer to the intersection points of the primary lines as grid vertices. There are two types of grid vertices: the grid vertices in $H$ are intersection of 3 primary lines and the ones outside $H$ are intersection of 2 primary lines.

Let $\sigma_0 = \sigma_0(m)$ and $\delta_0 = \delta_0(m)$ denote the basic parallelogram and triangle respectively, determined by the primary lines (i.e., lines in $L_2 \cup L_6 \cup L_{10}$) in all three possible orientations. The side length of $\sigma_0(m)$ and $\delta_0$ is $\frac{2}{m} \left(1 - O \left(\frac{1}{m}\right)\right)$. We refer to these basic parallelograms as grid cells. Recall that (i) the area of an equilateral triangle of side $s$ is $\frac{s^2 \sqrt{3}}{4}$; and (ii) the area of a regular hexagon of side $s$ is $\frac{s^2 \sqrt{3}}{2}$; as such, we have

$$\text{area}(H) = \frac{3 \sqrt{3}}{2},$$
$$\text{area}(\delta_0) = \frac{4 \sqrt{3}}{m^2} \left(1 - O \left(\frac{1}{m}\right)\right) = \frac{\sqrt{3}}{m^2} \left(1 - O \left(\frac{1}{m}\right)\right),$$
$$\text{area}(\sigma_0) = 2 \cdot \text{area}(\delta_0) = \frac{2 \sqrt{3}}{m^2} \left(1 - O \left(\frac{1}{m}\right)\right).$$

For $i = 3, \ldots, 12$, let $a_i$ denote the area of the (not necessarily connected) region
Figure 9: Detail of the construction with 12 slopes depicts the covering multiplicities inside the hexagon $H$. These numbers only reflect incidences at the grid vertices made by the primary lines.

covered by exactly $i$ of the 12 strips. Recall that $\text{area}(i, j, k)$ denotes the area of the triangle made by $\ell_i$, $\ell_j$ and $\ell_k$.

Observe that $a_{12}$ is the area of the 12-gon $\bigcap_{i=1}^{12} \Gamma_i$. This 12-gon is not regular, since consecutive vertices lie on two concentric cycles of radii $\frac{1}{3}$ and $\frac{\sqrt{3}}{2}$ centered at $(\frac{1}{2}, \frac{\sqrt{3}}{2})$. So $a_{12}$ is the sum of the areas of 12 congruent triangles; each with one vertex at the center of $H$ and other two as the two consecutive vertices of the 12-gon. Each of these triangles has area $\frac{\sqrt{3}}{60}$. Therefore,

$$a_{12} = 12 \cdot \frac{\sqrt{3}}{60} = \frac{\sqrt{3}}{5},$$

$$a_{11} = 12 \cdot \text{area}(1, 5, 9) = 12 \cdot \frac{1}{140\sqrt{3}} = \frac{\sqrt{3}}{35},$$

$$a_{10} = 6 \cdot (\text{area}(1, 5, 13) - \text{area}(1, 5, 9)) + 6 \cdot (\text{area}(5, 9, 22) - \text{area}(1, 5, 9)) = 6 \cdot \left(\frac{\sqrt{3}}{70} - \frac{1}{140\sqrt{3}}\right) + 6 \cdot \left(\frac{1}{56\sqrt{3}} - \frac{1}{140\sqrt{3}}\right) = \frac{13\sqrt{3}}{140}. $$
the relevant regions into account, we obtain

\[
a_9 = 12 \cdot (\text{area}(1, 7, 22) - \text{area}(1, 9, 22)) = 12 \cdot \left( \frac{1}{20\sqrt{3}} - \frac{1}{56\sqrt{3}} \right) = \frac{9\sqrt{3}}{70},
\]

\[
a_8 = 6 \cdot (\text{area}(9, 22, 24) - \text{area}(7, 22, 24)) + 12 \cdot \text{area}(7, 13, 22)
= 6 \cdot \left( \frac{\sqrt{3}}{40} - \frac{1}{20\sqrt{3}} \right) + 12 \cdot \frac{\sqrt{3}}{140} = \frac{19\sqrt{3}}{140},
\]

\[
a_7 = 12 \cdot (\text{area}(7, 22, 24) - \text{area}(13, 22, 24)) + 6 \cdot (\text{area}(1, 17, 22) - \text{area}(1, 13, 22))
= 12 \cdot \left( \frac{1}{20\sqrt{3}} - \frac{\sqrt{3}}{140} \right) + 6 \cdot \left( \frac{5}{28\sqrt{3}} - \frac{1}{14\sqrt{3}} \right) = \frac{23\sqrt{3}}{70},
\]

\[
a_6 = 12 \cdot (\text{area}(13, 22, 24)) + 6 \cdot (\text{area}(7, 11, 15) - 2 \cdot \text{area}(1, 11, 15))
= 12 \cdot \frac{\sqrt{3}}{140} + 6 \cdot \left( \frac{1}{4\sqrt{3}} - \frac{1}{20\sqrt{3}} \right) = \frac{27\sqrt{3}}{70},
\]

\[
a_5 = 12 \cdot (\text{area}(1, 11, 15)) + 6 \cdot (\text{area}(1, 11, 21)) = 12 \cdot \frac{1}{20\sqrt{3}} + 6 \cdot \frac{1}{4\sqrt{3}} = \frac{7\sqrt{3}}{10},
\]

\[
a_4 = 12 \cdot (\text{area}(1, 3, 11)) = 12 \cdot \frac{1}{4\sqrt{3}} = \sqrt{3},
\]

\[
a_3 = 12 \cdot (\text{area}(4, 7, 11)) = 12 \cdot \frac{\sqrt{3}}{4} = 3\sqrt{3}.
\]

The region whose area is \(\sum_{i=5}^{12} a_i\) consists of the hexagon \(H\) and 6 triangles outside \(H\). Therefore,

\[
\sum_{i=5}^{12} a_i = \text{area}(H) + 6 \cdot \text{area}(1, 11, 21) = \frac{3\sqrt{3}}{2} + 6 \cdot \frac{1}{4\sqrt{3}} = 2\sqrt{3}.
\]

Recall that \(\lambda_i(m)\) denotes the number of \(i\)-wise crossings where each bundle consists of \(m\) lines. Note that \(\lambda_i(m)\) is proportional to \(a_i\), for \(i = 5, 6, \ldots, 12\). Indeed, \(\lambda_i(m)\) is equal to the number of grid vertices that lie in a region covered by \(i\) strips, which is roughly equal to the ratio \(\frac{i}{\text{area}(\sigma_0)}\), for \(i = 5, 6, \ldots, 12\). More precisely, taking also the boundary effect of the relevant regions into account, we obtain

\[
\lambda_{12}(m) = \frac{a_{12}}{\text{area}(\sigma_0)} - O(m) = \frac{\sqrt{3}}{5} \frac{m^2}{2\sqrt{3}} - O(m) = \frac{m^2}{10} - O(m),
\]

\[
\lambda_{11}(m) = \frac{a_{11}}{\text{area}(\sigma_0)} - O(m) = \frac{m^2}{70} - O(m),
\]

\[
\lambda_{10}(m) = \frac{a_{10}}{\text{area}(\sigma_0)} - O(m) = \frac{13m^2}{280} - O(m),
\]

\[
\lambda_9(m) = \frac{a_9}{\text{area}(\sigma_0)} - O(m) = \frac{9m^2}{140} - O(m),
\]

\[
\lambda_8(m) = \frac{a_8}{\text{area}(\sigma_0)} - O(m) = \frac{19m^2}{280} - O(m),
\]

\[
\lambda_7(m) = \frac{a_7}{\text{area}(\sigma_0)} - O(m) = \frac{23m^2}{140} - O(m),
\]
\[
\lambda_0(m) = \frac{a_6}{\text{area}(\sigma_0)} - O(m) = \frac{27m^2}{140} - O(m),
\]
\[
\lambda_5(m) = \frac{a_6}{\text{area}(\sigma_0)} - O(m) = \frac{7m^2}{20} - O(m).
\]

For \(i = 3, 4\), not all the \(i\)-wise crossings are at grid vertices. It can be exhaustively verified (by hand) that there are 21 types of crossings; see Fig. 10. Types 1 through 3 are 4-wise crossings and types 4 through 21 are 3-wise crossings. The bundles intersecting at each of these 21 types of vertices are listed in Table 4. For \(j = 1, 2, \ldots, 21\), let \(w_j\) denote the weighted area containing all the crossings of type \(j\); where the weight is the number of crossings per grid cell. To complete the estimates of \(\lambda_i(m)\) for \(i = 3, 4\), we calculate \(w_j\) for all \(j\) from the bundles intersecting at type \(j\) crossings. The values are listed in Table 5. Observe that \(\Gamma_i \cap \Gamma_j\) is a parallelogram defined by the two pairs of parallel lines \(\ell_{2i-1}, \ell_{2i}\) and \(\ell_{2j-1}, \ell_{2j}\), respectively; thus \(\text{area}(\Gamma_i \cap \Gamma_j) = \text{area}(P(2i - 1, 2i, 2j - 1, 2j))\).

\[
\begin{array}{|c|c|c|}
\hline
j & \text{Bundles intersecting at type } j \text{ vertices} & j & \text{Bundles intersecting at type } j \text{ vertices} & j & \text{Bundles intersecting at type } j \text{ vertices} \\
\hline
1 & L_6, L_{12}, L_3, L_9 & 8 & L_4, L_{11}, L_9 & 15 & L_4, L_8, L_{12} \\
2 & L_2, L_8, L_{11}, L_5 & 9 & L_8, L_1, L_3 & 16 & L_6, L_{12}, L_3 \\
3 & L_{10}, L_4, L_1, L_7 & 10 & L_1, L_5, L_9 & 17 & L_6, L_{12}, L_9 \\
4 & L_2, L_7, L_9 & 11 & L_{11}, L_3, L_7 & 18 & L_2, L_8, L_{11} \\
5 & L_6, L_{11}, L_1 & 12 & L_{12}, L_3, L_9 & 19 & L_2, L_8, L_5 \\
6 & L_{10}, L_3, L_5 & 13 & L_4, L_1, L_7 & 20 & L_{10}, L_4, L_1 \\
7 & L_{12}, L_5, L_7 & 14 & L_8, L_{11}, L_5 & 21 & L_{10}, L_4, L_7 \\
\hline
\end{array}
\]

Table 4: Bundles intersecting at type \(j\) vertices for \(j = 1, 2, \ldots, 21\).

- To estimate \(\lambda_4(m)\), note that all the 4-wise crossings that are not at grid vertices, are at the centers of the grid cells; we have

\[
w_1 = \text{area}(\Gamma_6 \cap \Gamma_{12} \cap \Gamma_3 \cap \Gamma_9) = \text{area}(\Gamma_3 \cap \Gamma_9) = \text{area}(P(5, 6, 17, 18)) = \frac{\sqrt{3}}{4}.
\]

Types 2 and 3 are 120° and 240° rotations of type 1, respectively; therefore by symmetry,

\[
w_1 = w_2 = w_3 = \frac{\sqrt{3}}{4}.
\]

Consequently, we have

\[
\lambda_4(m) = \frac{a_4 + \sum_{j=1}^{3} w_j}{\text{area}(\sigma_0)} - O(m) = \left(\frac{1}{2} + \frac{3}{8}\right) m^2 - O(m) = \frac{7m^2}{8} - O(m).
\]

Lastly, we estimate \(\lambda_3(m)\). Besides 3-wise crossings at grid vertices in \(H\) (whose number is proportional to \(a_3\)), there are 18 types of 3-wise crossings i.e., types 4 through 21, on the boundary or in the interior of the grid cells in \(H\).
Figure 10: Types of incidences of 3 and 4 lines that are not at grid vertices: 4-wise crossings: types 1 through 3; 3-wise crossings: types 4 through 21. To list the coordinates of the crossing points (shown as blue dots), we set the leftmost vertex of the grid cell (shown in blue lines) at (0, 0) and the length of the sides of each grid cell as 1.

For types 1 through 3 the crossings are at the center of the parallelogram.

For types 4 through 6, the crossings are at $\frac{1}{3}\text{rd}$ and $\frac{2}{3}\text{rd}$ of the short diagonal.

For type 7, the crossings are at $(\frac{2}{5}, \frac{-\sqrt{3}}{10})$, $(\frac{2}{5}, \frac{-3\sqrt{3}}{10})$, $(1, \frac{-\sqrt{3}}{6})$, $(1, \frac{-2\sqrt{3}}{6})$.

For type 8, the crossings are at $(\frac{2}{5}, \frac{\sqrt{3}}{10})$, $(\frac{7}{10}, \frac{\sqrt{3}}{10})$, $(\frac{2}{5}, \frac{3\sqrt{3}}{5})$, $(\frac{11}{10}, \frac{3\sqrt{3}}{10})$.

For type 9, the crossings are at $(\frac{2}{5}, \frac{\sqrt{3}}{10})$, $(\frac{7}{10}, \frac{\sqrt{3}}{10})$, $(\frac{2}{5}, \frac{-\sqrt{3}}{5})$, $(\frac{7}{10}, \frac{-\sqrt{3}}{5})$.

For type 10, the crossings are at $(\frac{2}{5}, \frac{\sqrt{3}}{10})$, $(\frac{7}{10}, \frac{\sqrt{3}}{10})$, $(\frac{2}{5}, \frac{-\sqrt{3}}{5})$, $(\frac{7}{10}, \frac{-\sqrt{3}}{5})$.

For type 11, the crossings are at $(\frac{2}{5}, \frac{\sqrt{3}}{10})$, $(\frac{7}{10}, \frac{\sqrt{3}}{10})$, $(\frac{2}{5}, \frac{-\sqrt{3}}{5})$, $(\frac{7}{10}, \frac{-\sqrt{3}}{5})$.

For type 12, the crossings are at $(\frac{2}{5}, \frac{\sqrt{3}}{10})$ and $(1, \frac{-\sqrt{3}}{6})$.

For type 13, the crossings are at $(\frac{2}{5}, \frac{\sqrt{3}}{8})$ and $(\frac{5}{8}, \frac{3\sqrt{3}}{8})$.

For type 14, the crossings are at $(\frac{2}{5}, \frac{\sqrt{3}}{8})$ and $(\frac{5}{8}, \frac{-\sqrt{3}}{8})$.

For type 15, the crossings are at $\frac{1}{4}\text{rd}$ and $\frac{3}{4}\text{rd}$ of the long diagonal.

For types 16 through 21 the crossings are at the center of the parallelogram and can be obtained from types 1 through 3 by losing one of the tertiary bundles.

The relative positions of all these crossings are shown in Fig. 11.
• For types 4, 5, and 6, there are two crossings per grid cell; and

\[ w_4 = 2 \cdot \text{area}(\Gamma_2 \cap \Gamma_7 \cap \Gamma_9) = 2 \cdot (\text{area}(P(3, 4, 17, 18)) - \text{area}(3, 13, 17) - \text{area}(4, 14, 18)) = 2 \cdot \left( \frac{2}{\sqrt{3}} - \frac{1}{4\sqrt{3}} - \frac{1}{4\sqrt{3}} \right) = \sqrt{3}. \]

Types 5 and 6 are 120° and 240° rotations of type 4, respectively; therefore by symmetry,

\[ w_4 = w_5 = w_6 = \sqrt{3}. \]

• For types 7, 8, and 9, there are four crossings per grid cell; and

\[ w_7 = 4 \cdot \text{area}(\Gamma_{12} \cap \Gamma_5 \cap \Gamma_7) = 4 \cdot (\text{area}(P(9, 10, 13, 14)) - \text{area}(10, 13, 23) - \text{area}(9, 14, 24)) = 4 \cdot \left( \frac{2\sqrt{3}}{5} - \frac{\sqrt{3}}{20} - \frac{\sqrt{3}}{20} \right) = \frac{6\sqrt{3}}{5}. \]

Types 8 and 9 are 120° and 240° rotations of type 7, respectively; therefore by symmetry,

\[ w_7 = w_8 = w_9 = \frac{6\sqrt{3}}{5}. \]

• For types 10, 11, there are six crossings per grid cell; and

\[ w_{10} = 6 \cdot \text{area}(\Gamma_1 \cap \Gamma_5 \cap \Gamma_9) = 6 \cdot (\text{area}(P(1, 2, 17, 18)) - \text{area}(1, 9, 17) - \text{area}(2, 10, 18)) = 6 \cdot \left( \frac{2\sqrt{3}}{7} - \frac{\sqrt{3}}{28} - \frac{\sqrt{3}}{28} \right) = \frac{9\sqrt{3}}{7}. \]

Type 11 is the reflection in a vertical line of type 10; therefore by symmetry,

\[ w_{10} = w_{11} = \frac{9\sqrt{3}}{7}. \]

• For types 12, 13, and 14, there are two crossings per grid cell; and

\[ w_{12} = 2 \cdot \text{area}(\Gamma_{12} \cap \Gamma_3 \cap \Gamma_9) = 2 \cdot \text{area}(\Gamma_3 \cap \Gamma_9) = 2 \cdot \text{area}(P(5, 6, 17, 18)) = \frac{\sqrt{3}}{2}. \]

Types 13 and 14 are 120° and 240° rotations of type 12, respectively; therefore by symmetry,

\[ w_{12} = w_{13} = w_{14} = \frac{\sqrt{3}}{2}. \]
• For type 15, there are two crossings per grid cell; and
\[
    w_{15} = 2 \cdot \text{area}(\Gamma_4 \cap \Gamma_8 \cap \Gamma_{12})
    = 2 \cdot (\text{area}(P(15, 16, 23, 24)) - \text{area}(7, 15, 24) - \text{area}(8, 16, 23))
    = 2 \cdot \left( \frac{2}{\sqrt{3}} - \frac{1}{4\sqrt{3}} - \frac{1}{4\sqrt{3}} \right) = \sqrt{3}.
\]

• For types 16 through 21, there is one crossing per grid cell; and
\[
    w_{16} = \text{area}(\Gamma_6 \cap \Gamma_{12} \cap \Gamma_3 - \Gamma_9) = \text{area}(\Gamma_{12} \cap \Gamma_3) - \text{area}(\Gamma_{12} \cap \Gamma_3 \cap \Gamma_9)
    = \text{area}(P(5, 6, 23, 24)) - \text{area}(P(5, 6, 17, 18)) = \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{4}.
\]

Type 17 is the reflection in a vertical line of type 16, types 18 and 20 are 120° and 240° rotations of type 16, respectively. Types 19 and 21 are 120° and 240° rotations of type 17, respectively. Therefore by symmetry,
\[
    w_{16} = w_{17} = w_{18} = w_{19} = w_{20} = w_{21} = \frac{\sqrt{3}}{4}.
\]

Consequently, we have
\[
    \lambda_3(m) = \frac{a_3 + \sum_{j=4}^{21} w_j}{\text{area}(\sigma_0)} - O(m) = \left( \frac{3}{2} + \frac{3}{2} + \frac{9}{5} + \frac{9}{7} + \frac{3}{4} + \frac{1}{2} + \frac{3}{4} \right) m^2 - O(m)
    = \frac{283}{35} m^2 - O(m).
\]

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Table 5: Values of $w_j$ for $j = 1, \ldots, 21$.

The values of $\lambda_i(m)$, for $i = 3, \ldots, 12$, are summarized in Table 6; for convenience the linear terms are omitted. Since $m = n/12$, $\lambda_i$ can be also viewed as a function of $n$.

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<tr>
<td>$\lambda_i(m)$</td>
<td>$\frac{283m^2}{35}$</td>
<td>$\frac{7m^2}{8}$</td>
<td>$\frac{7m^2}{20}$</td>
<td>$\frac{27m^2}{140}$</td>
<td>$\frac{23m^2}{140}$</td>
<td>$\frac{19m^2}{280}$</td>
<td>$\frac{9m^2}{140}$</td>
<td>$\frac{13m^2}{280}$</td>
<td>$\frac{m^2}{70}$</td>
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<tr>
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<td>$\frac{7n^2}{8-144}$</td>
<td>$\frac{7n^2}{20-144}$</td>
<td>$\frac{27n^2}{140-144}$</td>
<td>$\frac{23n^2}{140-144}$</td>
<td>$\frac{19n^2}{280-144}$</td>
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<td>$\frac{13n^2}{280-144}$</td>
<td>$\frac{n^2}{70-144}$</td>
<td>$\frac{n^2}{10-144}$</td>
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</table>

Table 6: The asymptotic values of $\lambda_i(m)$ and $\lambda_i(n)$ for $i = 3, \ldots, 12$. 
The multiplicative factor in Eq. (4) is bounded from below as follows:

\[ F(n) \geq \prod_{i=3}^{12} B_i(n) \geq 2^{283n^2/35} \cdot 8^{7n^2/20} \cdot 62^{7n^2/20} \cdot 908^{27n^2/140} \cdot 24698^{23n^2/140} \]

\[ \cdot \ 1232944^{19n^2/280} \cdot 112018190^{9n^2/280} \cdot 18410581880^{13n^2/280} \]

\[ \cdot 5449192389984^{n^2/70}\cdot 2894710651370536^{n^2/10}\cdot 2^{-O(n)}. \]

Figure 11: In the 12-gon in the middle of \( H \), all the triangular grid cells contain 3-crossings and 4-crossings of all types 1 through 15. In other grid cells of the construction only some of these types appear.

We prove by induction on \( n \) that \( T(n) \geq 2^{cn^2-O(n \log n)} \) for a suitable constant \( c > 0 \). It suffices to choose \( c \) (using the values of \( B_i \) for \( i = 3, \ldots, 12 \) in Table 1) so that

\[
\frac{1}{144} \left( \frac{283}{35} + \frac{7}{8} \log 8 + \frac{7}{20} \log 62 + \frac{27}{140} \log 908 + \frac{23}{140} \log 24698 \right. \\
+ \frac{19}{280} \log 1232944 + \frac{9}{140} \log 112018190 + \frac{13}{280} \log 18410581880 \\
\left. + \frac{1}{70} \log 5449192389984 + \frac{1}{10} \log 2894710651370536 \right) \geq \frac{11c}{12}. 
\]

The above inequality holds if we set

\[
c = \frac{1}{132} \left( \frac{283}{35} + \frac{7}{8} \log 8 + \frac{7}{20} \log 62 + \frac{27}{140} \log 908 + \frac{23}{140} \log 24698 \\
+ \frac{19}{280} \log 1232944 + \frac{9}{140} \log 112018190 + \frac{13}{280} \log 18410581880 \\
+ \frac{1}{70} \log 5449192389984 + \frac{1}{10} \log 2894710651370536 \right) > 0.2083, \tag{7}
\]

and the lower bound in Theorem 1 follows.
4 Conclusion

We analyzed several recursive constructions derived from arrangements of lines with 3, 4, 6, 8, and 12 distinct slopes; in two different styles (rectangular and hexagonal). The hexagonal construction with 12 slopes yields the lower bound $b_n \geq 0.2083 n^2$ for large $n$. We think that increasing the number of slopes will further increase the lower bound, and likely the proof complexity at the same time. The questions of how far can one go and whether there are other more efficient variants remain. We conclude with the following questions.

1. What lower bounds on $B_n$ can be deduced from line arrangements with a higher number of slopes? In particular, hexagonal and rectangular constructions with 16 slopes seem to be the most promising candidates. Note that the value of $B_{16}$ is currently unknown.

2. What lower bounds on $B_n$ can be obtained from rhombic tilings of a centrally symmetric octagon? Or from those of a centrally symmetric $k$-gon for some other even $k \geq 10$? No closed formulas for the number of such tilings seem to be available at the time of this writing. However, suitable estimates could perhaps be deduced from previous results; see, e.g., [1, 2, 4, 12].

References


A Rectangular construction with 8 slopes

We describe and analyze a rectangular construction with lines of 8 slopes. See Fig. 12. Consider 8 bundles of parallel lines whose slopes are 0, ∞, ±1/2, ±1, ±2. The axes of all strips are incident to the center of $U$. This construction yields the lower bound $b_n \geq 0.1999 \cdot n^2$ for large $n$.

Let $\mathcal{L} = \mathcal{L}_1 \cup \ldots \cup \mathcal{L}_8$ be the partition of the lines into eight bundles of parallel lines. The $m$ lines in $\mathcal{L}_i$ are contained in the strip $\Gamma_i$ bounded by the two lines $\ell_{2i-1}$ and $\ell_{2i}$, for $i = 1, \ldots, 8$. The equation of line $\ell_i$ is $\alpha_i x + \beta_i y + \gamma_i = 0$, with $\alpha_i, \beta_i, \gamma_i$, for $i = 1, \ldots, 16$ given in Fig. 13 (right). Observe that $U = \Gamma_4 \cap \Gamma_8$.

We refer to lines in $\mathcal{L}_4 \cup \mathcal{L}_8$ (i.e., axis-aligned lines) as the primary lines, and to rest of the lines as secondary lines. We refer to the intersection points of the primary lines as grid vertices. The slopes of the primary lines are in $\{0, \infty\}$. The slopes of the secondary lines are in $\{\pm 1/2, \pm 1, \pm 2\}$. Note that the distance between consecutive lines

- in $\mathcal{L}_4$ or $\mathcal{L}_8$ is $\frac{1}{m} \left( 1 - O\left( \frac{1}{m} \right) \right)$;
- in $\mathcal{L}_2$ or $\mathcal{L}_6$ is $\frac{1}{m\sqrt{2}} \left( 1 - O\left( \frac{1}{m} \right) \right)$;
- in $\mathcal{L}_1$, $\mathcal{L}_3$, $\mathcal{L}_5$, or $\mathcal{L}_7$ is $\frac{1}{m\sqrt{3}} \left( 1 - O\left( \frac{1}{m} \right) \right)$.

Let $\sigma_0 = \sigma_0(m)$ denote the basic parallelogram (here, square) determined by consecutive axis-aligned lines (i.e., lines in $\mathcal{L}_4 \cup \mathcal{L}_8$); the side length of $\sigma_0$ is $\frac{1}{m} \left( 1 - O\left( \frac{1}{m} \right) \right)$. 

Figure 12: Construction with 8 slopes.
We refer to these basic parallelograms as *grid cells*. Let \( U' \) be the smaller square made by \( \ell_5, \ell_6, \ell_{13}, \ell_{14}, \) i.e., \( U' = \Gamma_3 \cap \Gamma_7 \); the similarity ratio \( \rho(U', U) \) is equal to \( \frac{1}{\sqrt{5}} \). We have

\[
\begin{align*}
\text{area}(U) &= 1, \\
\text{area}(U') &= \frac{\text{area}(U)}{5} = \frac{1}{5}, \\
\text{area}(\sigma_0) &= \frac{1}{m^2} \left( 1 - O \left( \frac{1}{m} \right) \right).
\end{align*}
\]

For \( i = 3, 4, \ldots, 8 \), let \( a_i \) denote the area of the (not necessarily connected) region covered by exactly \( i \) of the 8 strips. Recall that \( \text{area}(i,j,k) \) denotes the area of the triangle made by \( \ell_i, \ell_j \) and \( \ell_k \). We have

\[
\begin{align*}
a_3 &= 8 \cdot \text{area}(3,7,15) = 1, \\
a_4 &= 8 \cdot \text{area}(5,7,11) = \frac{1}{3}, \\
a_5 &= 4 \left( 2 \cdot \text{area}(5,11,13) + \text{area}(2,5,11) \right) = \frac{7}{30}, \\
a_6 &= 4 \left( \text{area}(6,11,13) - 2 \cdot \text{area}(2,11,9) - \text{area}(2,9,13) \right) = \frac{1}{5}, \\
a_7 &= 8 \cdot \text{area}(5,9,13) = \frac{1}{15},
\end{align*}
\]

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</table>

Figure 13: Left: The eight strips and the corresponding covering multiplicities. These numbers only reflect incidences at the grid vertices made by axis-aligned lines. Right: Coefficients of the lines \( \ell_i \) for \( i = 1,2, \ldots, 16 \).
\[ a_8 = \text{area}(U') - 4 \cdot \text{area}(5, 9, 13) = \frac{1}{5} - \frac{1}{30} = \frac{1}{6}. \]

Observe that \( a_4 + a_5 + a_6 + a_7 + a_8 = \text{area}(U) = 1. \) Recall that \( \lambda_i(m) \) denote the number of \( i \)-wise crossings where each bundle consists of \( m \) lines. Note that \( \lambda_i(m) \) is proportional to \( a_i \), for \( i = 4, 5, 6, 7, 8 \). Indeed, \( \lambda_i(m) \) is equal to the number of grid points that lie in a region covered by \( i \) strips, which is roughly equal to the ratio \( \frac{a_i}{\text{area}(\sigma_0)} \), for \( i = 4, 5, 6, 7, 8 \). More precisely, taking also the boundary effect of the relevant regions into account, we obtain

\[
\lambda_4(m) = \frac{a_4}{\text{area}(\sigma_0)} - O(m) = \frac{m^2}{3} - O(m),
\]
\[
\lambda_5(m) = \frac{a_5}{\text{area}(\sigma_0)} - O(m) = \frac{7m^2}{30} - O(m),
\]
\[
\lambda_6(m) = \frac{a_6}{\text{area}(\sigma_0)} - O(m) = \frac{m^2}{5} - O(m),
\]
\[
\lambda_7(m) = \frac{a_7}{\text{area}(\sigma_0)} - O(m) = \frac{m^2}{15} - O(m),
\]
\[
\lambda_8(m) = \frac{a_8}{\text{area}(\sigma_0)} - O(m) = \frac{m^2}{6} - O(m).
\]

For estimating \( \lambda_3(m) \), in addition to considering 3-wise crossings in the exterior of \( U \), we also observe 3-wise crossings on the boundaries or in the interior of the small grid cells contained in some regions of \( U \). Specifically, we distinguish exactly four types of 3-wise crossings, as illustrated and specified in Fig. 14. For \( j = 1, 2, 3, 4 \), let \( w_j \) denote the weighted area containing all crossings of type \( j \), where the weight is the number of 3-wise crossings per grid cell. To complete the estimate of \( \lambda_3(m) \), we calculate \( w_j \) for all \( j \), from the bundles intersecting at crossings of type \( j \); listed in Fig. 14 (right).

![Figure 14: Left: Other types of 3-wise crossings. Right: Intersecting bundles for these crossings.](image)

Observe that \( \Gamma_i \cap \Gamma_j \) is a parallelogram defined by the two pairs of parallel lines \( \ell_{2i-1}, \ell_{2i} \) and \( \ell_{2j-1}, \ell_{2j} \), respectively, thus \( \text{area}(\Gamma_i \cap \Gamma_j) = \text{area}(P(2i - 1, 2i, 2j - 1, 2j)) \). For types 1 and 2, there is one crossing per grid cell and for types 3 and 4, there are two crossings per grid cell. Therefore we have,

\[
\begin{align*}
w_1 &= \text{area}(\Gamma_4 \cap \Gamma_1 \cap \Gamma_7) = \text{area}(\Gamma_1 \cap \Gamma_7) = \text{area}(P(1, 2, 13, 14)) = 1/4, \\
w_2 &= \text{area}(\Gamma_8 \cap \Gamma_3 \cap \Gamma_5) = \text{area}(\Gamma_3 \cap \Gamma_5) = \text{area}(P(5, 6, 9, 10)) = 1/4.
\end{align*}
\]
\[ w_3 = 2 \cdot \text{area}(\Gamma_1 \cap \Gamma_3 \cap \Gamma_6) = 2 \cdot (\text{area}(P(1, 2, 5, 6)) - 2 \cdot \text{area}(2, 5, 11)) = 2 \cdot (1/3 - 1/12) = 1/2, \]

\[ w_4 = 2 \cdot \text{area}(\Gamma_2 \cap \Gamma_5 \cap \Gamma_7) = 1/2. \]

It follows that
\[
\lambda_3(m) = \frac{a_3 + \sum_{j=1}^{4} w_j}{\text{area}(\sigma_0)} - O(m) = \left(1 + \frac{1}{4} + \frac{1}{4} + \frac{1}{2} + \frac{1}{2}\right) m^2 - O(m) = \frac{5m^2}{2} - O(m).
\]

The values of \( \lambda_i(m) \), for \( i = 3, 4, \ldots, 8 \), are summarized in Table 7; for convenience the linear terms are omitted. Since \( m = n/8 \), \( \lambda_i \) can be also viewed as a function of \( n \).

<table>
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<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
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<td>( \lambda_i(m) )</td>
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</table>

Table 7: The asymptotic values of \( \lambda_i(m) \) and \( \lambda_i(n) \) for \( i = 3, 4, \ldots, 8 \).

The multiplicative factor in Eq. (4) is bounded from below as follows:
\[
F(n) \geq \prod_{i=3}^{8} B_i^{\lambda_i(n)} \geq 2^{\frac{5n^2}{2}} \cdot \frac{8n^2}{32} \cdot \frac{7n^2}{320} \cdot \frac{908n^2}{8} \cdot \frac{24698n^2}{15} \cdot \frac{1232944n^2}{6} \cdot \frac{2}{O(n)}.
\]

We prove by induction on \( n \) that \( T(n) \geq 2^{cn^2-O(n \log n)} \) for a suitable constant \( c > 0 \). It suffices to choose \( c \) (using the values of \( B_i \) for \( i = 3, \ldots, 8 \) in Table 1) so that
\[
\frac{1}{64} \left(\frac{5}{2} + \frac{1}{3} \log 8 + \frac{7}{30} \log 62 + \frac{1}{5} \log 908 + \frac{1}{15} \log 24698 + \frac{1}{6} \log 1232944\right) \geq \frac{7c}{8}.
\]

The above inequality holds if we set
\[
c = \frac{1}{56} \left(\frac{5}{2} + 1 + \frac{7}{30} \log 62 + \frac{1}{5} \log 908 + \frac{1}{15} \log 24698 + \frac{1}{6} \log 1232944\right) > 0.1999, \quad (8)
\]

and this yields the lower bound \( B_n \geq 2^{cn^2-O(n \log n)} \), for some constant \( c > 0.1999 \). In particular, we have \( B_n \geq 2^{0.1999 n^2} \) for large \( n \).

**B Rectangular construction with 12 slopes**

We next describe and analyze a rectangular construction with lines of 12 slopes. Consider 12 bundles of parallel lines whose slopes are 0, \( \infty \), \( \pm 1/3 \), \( \pm 1/2 \), \( \pm 1 \), \( \pm 2 \), \( \pm 3 \). The axes of all strips are incident to the center of \( U = [0, 1]^2 \). Refer to Fig. 15. This construction yields the lower bound \( b_n \geq 0.2053 n^2 \) for large \( n \).
Let $\mathcal{L} = \mathcal{L}_1 \cup \ldots \cup \mathcal{L}_{12}$ be the partition of the lines into twelve bundles of parallel lines. The $m$ lines in $\mathcal{L}_i$ are contained in the strip $\Gamma_i$ bounded by the two lines $\ell_{2i-1}$ and $\ell_{2i}$, for $i = 1, \ldots, 12$. The equation of line $\ell_i$ is $\alpha_i x + \beta_i y + \gamma_i = 0$, with $\alpha_i, \beta_i, \gamma_i$, for $i = 1, \ldots, 24$ given in Table 8. Observe that $U = \Gamma_6 \cap \Gamma_{12}$.

We refer to lines in $\mathcal{L}_6 \cup \mathcal{L}_{12}$ (i.e., axis-aligned lines) as the primary lines, and to rest of the lines as the secondary lines. We refer to the intersection points of the primary lines as

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Figure 15: Construction with 12 slopes. The twelve strips and the corresponding covering multiplicities. These numbers only reflect incidences at the grid vertices made by axis-aligned lines.
grid vertices. The slopes of the primary lines are in \{0, \infty\}, and the slopes of the secondary lines are in \{\pm 1/3, \pm 1/2, \pm 1, \pm 2, \pm 3\}. Note that the distance between consecutive lines

- in \(L_6\) or \(L_{12}\) is \(\frac{1}{m} \left(1 - O\left(\frac{1}{m}\right)\right)\);
- in \(L_3\) or \(L_9\) is \(\frac{1}{m\sqrt{2}} \left(1 - O\left(\frac{1}{m}\right)\right)\);
- in \(L_2, L_4, L_8\), or \(L_{10}\) is \(\frac{1}{m\sqrt{5}} \left(1 - O\left(\frac{1}{m}\right)\right)\);
- in \(L_1, L_5, L_7\), or \(L_{11}\) is \(\frac{1}{m\sqrt{10}} \left(1 - O\left(\frac{1}{m}\right)\right)\).

Let \(\sigma_0 = \sigma_0(m)\) denote the basic parallelogram (here, square) determined by consecutive axis-aligned lines (i.e., lines in \(L_6 \cup L_{12}\)); the side length of \(\sigma_0\) is \(\frac{1}{m} \left(1 - O\left(\frac{1}{m}\right)\right)\). We refer to these basic parallelograms as grid cells. Let \(U_1 = \Gamma_1 \cap \Gamma_7\), be the square made by \(\ell_1, \ell_2, \ell_{13}, \ell_{14}\), and let \(U_2 = \Gamma_2 \cap \Gamma_8\), be the smaller square made by \(\ell_3, \ell_4, \ell_{15}, \ell_{16}\). Note that \(\rho(U_1, U) = \frac{1}{\sqrt{10}}\) and \(\rho(U_2, U) = \frac{1}{\sqrt{5}}\). We also have

\[
\begin{align*}
\text{area}(U) &= 1, \\
\text{area}(U_1) &= \frac{\text{area}(U)}{10} = \frac{1}{10}, \\
\text{area}(U_2) &= \frac{\text{area}(U)}{5} = \frac{1}{5}, \\
\text{area}(\sigma_0) &= \frac{1}{m^2} \left(1 - O\left(\frac{1}{m}\right)\right).
\end{align*}
\]

For \(i = 3, \ldots, 12\), let \(a_i\) denote the area of the (not necessarily connected) region covered by exactly \(i\) of the 12 strips. Recall that \(\text{area}(i,j,k)\) denotes the area of the triangle bounded by \(\ell_i, \ell_j\) and \(\ell_k\). We have

\[
\begin{align*}
a_3 &= 8 \cdot (\text{area}(2, 11, 13) + \text{area}(3, 5, 23)) = 8 \cdot \left(\frac{1}{8} + \frac{1}{24}\right) = \frac{4}{3}, \\
a_4 &= 8 \cdot (\text{area}(2, 5, 11) + \text{area}(2, 7, 11)) = 8 \cdot \left(\frac{1}{12} + \frac{1}{120}\right) = \frac{11}{15}.
\end{align*}
\]

### Table 8: Coefficients of the 24 lines.

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<td>22</td>
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<td>1</td>
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</tr>
<tr>
<td>23</td>
<td>-1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>24</td>
<td>-1</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
\[ a_5 = 4 \cdot (\text{area}(11, 17, 23) - 2 \cdot \text{area}(2, 7, 11) - 2 \cdot \text{area}(2, 7, 17)) \]
\[ = 4 \left( \frac{1}{8} - \frac{2}{120} - \frac{2}{120} \right) = \frac{11}{30}, \]
\[ a_6 = 4 \cdot (2 \cdot \text{area}(7, 17, 19) + 2 \cdot \text{area}(2, 7, 17)) = 4 \cdot \left( \frac{2}{120} + \frac{2}{120} \right) = \frac{2}{15}, \]
\[ a_7 = 4 \cdot (2 \cdot \text{area}(7, 19, 21) + 2 \cdot (\text{area}(9, 17, 19) - \text{area}(7, 17, 19))) \]
\[ = 4 \cdot \left( \frac{1}{140} + 2 \cdot \left( \frac{1}{56} - \frac{1}{120} \right) \right) = \frac{11}{105}, \]
\[ a_8 = 8 \cdot (\text{area}(9, 19, 21) - \text{area}(7, 19, 21)) + 4 \cdot (\text{area}(2, 9, 15) - \text{area}(9, 15, 19)) \]
\[ + 8 \cdot \text{area}(7, 21, 25) = \frac{13}{105}, \]
\[ a_9 = 8 \cdot (\text{area}(7, 15, 21) + \text{area}(9, 15, 19)) = 8 \cdot \left( \frac{1}{280} + \frac{1}{840} \right) = \frac{4}{105}, \]
\[ a_{10} = 4 \cdot ((\text{area}(7, 13, 15) - \text{area}(9, 13, 15)) + (\text{area}(13, 19, 21) - \text{area}(15, 19, 21))) \]
\[ = 4 \cdot \left( \left( \frac{1}{40} - \frac{1}{60} \right) + \left( \frac{1}{80} - \frac{1}{120} \right) \right) = 4 \cdot \left( \frac{1}{120} + \frac{1}{240} \right) = \frac{1}{20}, \]
\[ a_{11} = 8 \cdot \text{area}(2, 13, 21) = \frac{8}{240} = \frac{1}{30}, \]
\[ a_{12} = \text{area}(U_1) - 4 \cdot \text{area}(9, 13, 21) = \frac{1}{10} - \frac{4}{240} = \frac{1}{12}. \]

The region whose area is \( \sum_{i=4}^{12} a_i \) consists of \( U \) and 8 triangles outside \( U \). Therefore,
\[ \sum_{i=4}^{12} a_i = \text{area}(U) + 8 \cdot \text{area}(2, 5, 11) = 1 + 2/3 = 5/3. \]

Recall that \( \lambda_i(m) \) denote the number of \( i \)-wise crossings where each bundle consists of \( m \) lines. Note that \( \lambda_i(m) \) is proportional to \( a_i \), for \( i = 7, 8, \ldots, 12 \). Indeed, \( \lambda_i(m) \) is equal to the number of grid vertices, i.e., intersection points of the axis-parallel lines that lie in a region covered by \( i \) strips, which is roughly equal to the ratio \( \frac{a_i}{\text{area}(\sigma_0)} \), for \( i = 7, 8, \ldots, 12 \). More precisely, taking also the boundary effect of the relevant regions into account, we obtain
\[ \lambda_7(m) = \frac{a_7}{\text{area}(\sigma_0)} - O(m) = \frac{11m^2}{105} - O(m), \]
\[ \lambda_8(m) = \frac{a_8}{\text{area}(\sigma_0)} - O(m) = \frac{13m^2}{105} - O(m), \]
\[ \lambda_9(m) = \frac{a_9}{\text{area}(\sigma_0)} - O(m) = \frac{4m^2}{105} - O(m), \]
\[ \lambda_{10}(m) = \frac{a_{10}}{\text{area}(\sigma_0)} - O(m) = \frac{m^2}{20} - O(m), \]
\[ \lambda_{11}(m) = \frac{a_{11}}{\text{area}(\sigma_0)} - O(m) = \frac{m^2}{30} - O(m), \]
\[
\lambda_{12}(m) = \frac{a_{12}}{\text{area}(\sigma_0)} - O(m) = \frac{m^2}{12} - O(m).
\]

For \(i = 3, 4, 5, 6\), not all the \(i\)-wise crossings are at grid vertices. It can be exhaustively verified (by hand) that there are 29 types of such crossings in total; see Fig. 16. The bundles intersecting at each of these 29 types of vertices are listed in Table 9. For \(j = 1, 2, \ldots, 29\), let \(w_j\) denote the weighted area containing all crossings of type \(j\); where the weight is the number of crossings per grid cell. To complete the estimates of \(\lambda_i(m)\) for \(i = 3, 4, 5, 6\), we calculate \(w_j\) for all \(j\) from the bundles intersecting at type \(j\) crossings. The values are listed in Table 10.

<table>
<thead>
<tr>
<th>(j)</th>
<th>Bundles intersecting at type (j) vertices</th>
<th>(j)</th>
<th>Bundles intersecting at type (j) vertices</th>
<th>(j)</th>
<th>Bundles intersecting at type (j) vertices</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(\mathcal{L}_2, \mathcal{L}<em>6, \mathcal{L}</em>{10})</td>
<td>11 &amp; 12</td>
<td>(\mathcal{L}_1, \mathcal{L}<em>6, \mathcal{L}</em>{11})</td>
<td>21</td>
<td>(\mathcal{L}_3, \mathcal{L}_7, \mathcal{L}<em>9, \mathcal{L}</em>{11})</td>
</tr>
<tr>
<td>2</td>
<td>(\mathcal{L}_4, \mathcal{L}<em>8, \mathcal{L}</em>{12})</td>
<td>13</td>
<td>(\mathcal{L}_2, \mathcal{L}<em>8, \mathcal{L}</em>{11})</td>
<td>22</td>
<td>(\mathcal{L}_1, \mathcal{L}_3, \mathcal{L}_5, \mathcal{L}_9)</td>
</tr>
<tr>
<td>3</td>
<td>(\mathcal{L}_2, \mathcal{L}_4, \mathcal{L}_9)</td>
<td>14</td>
<td>(\mathcal{L}_1, \mathcal{L}<em>4, \mathcal{L}</em>{10})</td>
<td>23</td>
<td>(\mathcal{L}_1, \mathcal{L}_4, \mathcal{L}<em>7, \mathcal{L}</em>{10})</td>
</tr>
<tr>
<td>4</td>
<td>(\mathcal{L}_3, \mathcal{L}<em>8, \mathcal{L}</em>{10})</td>
<td>15</td>
<td>(\mathcal{L}_2, \mathcal{L}_5, \mathcal{L}_8)</td>
<td>24</td>
<td>(\mathcal{L}_2, \mathcal{L}_5, \mathcal{L}<em>8, \mathcal{L}</em>{11})</td>
</tr>
<tr>
<td>5</td>
<td>(\mathcal{L}_7, \mathcal{L}_9)</td>
<td>16</td>
<td>(\mathcal{L}_1, \mathcal{L}<em>7, \mathcal{L}</em>{10})</td>
<td>25</td>
<td>(\mathcal{L}_1, \mathcal{L}_3, \mathcal{L}<em>7, \mathcal{L}</em>{11})</td>
</tr>
<tr>
<td>6</td>
<td>(\mathcal{L}_3, \mathcal{L}_5, \mathcal{L}_9)</td>
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<td>(\mathcal{L}_1, \mathcal{L}_5, \mathcal{L}_9)</td>
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<tr>
<td>7</td>
<td>(\mathcal{L}_3, \mathcal{L}<em>9, \mathcal{L}</em>{11})</td>
<td>18</td>
<td>(\mathcal{L}_3, \mathcal{L}<em>7, \mathcal{L}</em>{11})</td>
<td>27</td>
<td>(\mathcal{L}_3, \mathcal{L}_5, \mathcal{L}<em>7, \mathcal{L}</em>{11})</td>
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<tr>
<td>8</td>
<td>(\mathcal{L}_1, \mathcal{L}_3, \mathcal{L}_9)</td>
<td>19</td>
<td>(\mathcal{L}_3, \mathcal{L}_5, \mathcal{L}_7, \mathcal{L}_9)</td>
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<td>(\mathcal{L}_1, \mathcal{L}_3, \mathcal{L}_5, \mathcal{L}_7, \mathcal{L}_9)</td>
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<tr>
<td>9 &amp; 10</td>
<td>(\mathcal{L}_5, \mathcal{L}<em>7, \mathcal{L}</em>{12})</td>
<td>20</td>
<td>(\mathcal{L}_1, \mathcal{L}_3, \mathcal{L}<em>9, \mathcal{L}</em>{11})</td>
<td>29</td>
<td>(\mathcal{L}_1, \mathcal{L}_3, \mathcal{L}_5, \mathcal{L}_7, \mathcal{L}<em>9, \mathcal{L}</em>{11})</td>
</tr>
</tbody>
</table>

Table 9: Bundles intersecting at type \(j\) vertices for \(j = 1, 2, \ldots, 29\).

For \(\lambda_6(m)\), all the 6-wise crossings that are not at grid vertices, are at the centers of grid cells; we have

\[
w_{29} = \text{area}(\Gamma_1 \cap \Gamma_3 \cap \Gamma_5 \cap \Gamma_7 \cap \Gamma_9 \cap \Gamma_{11}) = \text{area}(\Gamma_1 \cap \Gamma_5 \cap \Gamma_7 \cap \Gamma_{11}) = a_{12}.
\]

It follows that

\[
\lambda_6(m) = \frac{a_6 + w_{29}}{\text{area}(\sigma_0)} - O(m) = \frac{a_6 + a_{12}}{\text{area}(\sigma_0)} - O(m) = \frac{2m^2}{15} + \frac{m^2}{12} - O(m) = \frac{13m^2}{60} - O(m).
\]

Similarly for \(\lambda_5(m)\), all the 5-wise crossings that are not at grid vertices, i.e., types 25 through 28, are in the interiors of grid cells contained in eight small triangles inside \(U\). For example,

\[
w_{28} = \text{area}(\Gamma_1 \cap \Gamma_3 \cap \Gamma_5 \cap \Gamma_7 \cap \Gamma_9 - \Gamma_{11}) = \text{area}(1, 14, 22) + \text{area}(2, 13, 21) = 1/120.
\]

Observe that sum of the areas of these eight small triangles equals to \(a_{11}\). It follows that

\[
\lambda_5(m) = \frac{a_5 + \sum_{j=25}^{28} w_j}{\text{area}(\sigma_0)} - O(m) = \frac{a_5 + a_{11}}{\text{area}(\sigma_0)} - O(m) = \frac{11m^2}{30} + \frac{m^2}{30} - O(m) = \frac{2m^2}{5} - O(m).
\]
Figure 16: Types of incidences of 3, 4, 5, and 6 lines that are not at grid vertices: 3-wise crossings: types 1 through 18; 4-wise crossings: types 19 through 24; 5-wise crossings: types 25 through 28; 6-wise crossings: type 29.

For some types, the crossings are in the middle of a grid cell. To list the coordinates of crossing points, we rescale the grid cells to the unit square \([0,1]^2\).

For types 1 and 2, the crossings are at the midpoint of the horizontal and the vertical grid edges respectively. For type 3, the crossings are at \((1/3,1/3)\) and \((2/3,2/3)\).

For type 4, the crossings are at \((1/3,2/3)\) and \((2/3,1/3)\).

For types 9 and 10, the crossings are on vertical grid edges at height 1/3 and 2/3 from the horizontal line below, respectively.

For types 11 and 12, the crossings are on horizontal grid edges at distance 1/3 and 2/3 from the vertical line on the left, respectively.

For type 13, the crossings are at \((1/5,3/5)\) and \((3/5,4/5)\) and \((4/5,2/5)\) and \((2/5,1/5)\).

For type 14, the crossings are at \((1/5,2/5)\) and \((2/5,4/5)\) and \((4/5,3/5)\) and \((3/5,1/5)\).

For type 15, the crossings are at \((1/5,3/5)\) and \((3/5,4/5)\) and \((4/5,2/5)\) and \((2/5,1/5)\).

For type 16, the crossings are at \((1/5,2/5)\) and \((2/5,4/5)\) and \((4/5,3/5)\) and \((3/5,1/5)\).

For type 17, the crossings are at \((1/4,1/4)\) and \((3/4,3/4)\).

For type 18, the crossings are at \((1/4,3/4)\) and \((3/4,1/4)\).

For type 23, the crossings are at \((1/5,2/5)\) and \((2/5,4/5)\) and \((4/5,3/5)\) and \((3/5,1/5)\).

For type 24, the crossings are at \((1/5,3/5)\) and \((3/5,4/5)\) and \((4/5,2/5)\) and \((2/5,1/5)\).

For the other types, the crossings are at \((1/2,1/2)\).

Figure 17: These incidence patterns cannot occur.
To estimate $\lambda_4(m)$, note that besides 4-wise crossings at grid vertices, there are six types of 4-wise crossings i.e., types 19 through 24, in the interiors of grid cells:

- For types 19 and 20, there is one crossing per grid cell; and
  
  $$w_{19} = \text{area}(\Gamma_3 \cap \Gamma_5 \cap \Gamma_7 \cap \Gamma_9 - \Gamma_1 - \Gamma_{11})$$
  $$= (\text{area}(2, 10, 13) - \text{area}(2, 10, 21)) + (\text{area}(9, 14, 22) - \text{area}(1, 14, 22)) = 1/15.$$  
  Type 20 is a $90^\circ$ rotation of type 19; therefore by symmetry,
  $$w_{19} = w_{20} = 1/15.$$

- For types 21 and 22, there is one crossing per grid cell; and
  
  $$w_{21} = \text{area}(\Gamma_3 \cap \Gamma_7 \cap \Gamma_9 \cap \Gamma_{11} - \Gamma_1 - \Gamma_5)$$
  $$= (\text{area}(2, 14, 21) - \text{area}(2, 10, 21)) + (\text{area}(1, 13, 22) - \text{area}(1, 9, 22)) = 1/40.$$  
  Type 22 is the reflection in a vertical line of type 21; therefore by symmetry,
  $$w_{21} = w_{22} = 1/40.$$

- For types 23 and 24, there are four crossings per grid cell; and
  
  $$w_{23} = 4 \cdot \text{area}(\Gamma_1 \cap \Gamma_4 \cap \Gamma_7 \cap \Gamma_{10}) = 4 \cdot \text{area}(\Gamma_1 \cap \Gamma_7) = 4 \cdot \text{area}(U_1) = 2/5.$$  
  Type 24 is the reflection in a vertical line of type 23; therefore by symmetry,
  $$w_{23} = w_{24} = 2/5.$$  

Consequently, we have

$$\lambda_4(m) = \frac{a_4 + \sum_{j=19}^{24} w_j}{\text{area}(\sigma_0)} - O(m) = \left(\frac{11}{15} + \frac{2}{15} + \frac{1}{20} + \frac{4}{5}\right)m^2 - O(m) = \frac{103m^2}{60} - O(m).$$  

Lastly, we estimate $\lambda_3(m)$. Besides 3-wise crossings at grid vertices, there are 18 types of 3-wise crossings i.e., types 1 through 18, in the interior of grid cells:

- For types 1 and 2, there is one crossing per grid cell; and
  
  $$w_1 = \text{area}(\Gamma_2 \cap \Gamma_6 \cap \Gamma_{10}) = \text{area}(\Gamma_2 \cap \Gamma_{10}) = \text{area}(P(3, 4, 19, 20)) = 1/4.$$  
  Type 2 is a $90^\circ$ rotation of type 1; therefore by symmetry,
  $$w_1 = w_2 = 1/4.$$

- For types 3 and 4, there are two crossings per grid cell; and
  
  $$w_3 = 2 \cdot (\text{area}(\Gamma_2 \cap \Gamma_4 \cap \Gamma_9)) = 2 \cdot (\text{area}(P(3, 4, 7, 8)) - \text{area}(3, 8, 18) - \text{area}(4, 7, 17)) = 1/2.$$  
  Type 4 is the reflection in a vertical line of type 3; therefore by symmetry,
  $$w_3 = w_4 = 1/2.$$
• For types 5, 6, 7, 8, there is one crossing per grid cell; and
  \[ w_5 = \text{area}(\Gamma_3 \cap \Gamma_7 \cap \Gamma_9 - \Gamma_1 - \Gamma_5 - \Gamma_{11}) = \text{area}(5, 9, 22) + \text{area}(6, 10, 21)) = 1/20. \]

Type 6 is the reflection in a vertical line of type 5, and types 7 and 8 are 90\(^\circ\) rotations of types 6 and 5, respectively. Therefore by symmetry,
  \[ w_5 = w_6 = w_7 = w_8 = 1/20. \]

• For types 9, 10, 11, 12, there is one crossing on the boundary of each grid cell; and
  \[ w_9 = \text{area}(\Gamma_5 \cap \Gamma_7 \cap \Gamma_{12}) = \text{area}(\Gamma_5 \cap \Gamma_7) = \text{area}(P(9, 10, 13, 14)) = 1/6. \]

Type 10 is the reflection in a horizontal line of type 9, and types 11 and 12 are 90\(^\circ\) rotations of types 9 and 10, respectively. Therefore by symmetry,
  \[ w_9 = w_{10} = w_{11} = w_{12} = 1/6. \]

• For types 13, 14, 15, 16, there are four crossings per grid cell; and
  \[ w_{13} = 4 \cdot (\text{area}(\Gamma_2 \cap \Gamma_8 \cap \Gamma_{11} - \Gamma_5)) = 4 \cdot (\text{area}(3, 9, 13) + \text{area}(4, 10, 16)) = 1/5. \]

Type 14 is the reflection in a vertical line of type 13, and types 15 and 16 are 90\(^\circ\) rotations of types 13 and 14, respectively. Therefore by symmetry,
  \[ w_{13} = w_{14} = w_{15} = w_{16} = 1/5. \]

• For types 17 and 18, there are two crossings per grid cell; and
  \[ w_{17} = 2 \cdot (\text{area}(\Gamma_1 \cap \Gamma_5 \cap \Gamma_9)) = 2 \cdot (\text{area}(\Gamma_1 \cap \Gamma_5)) = 2 \cdot \text{area}(P(1, 2, 9, 10)) = 1/4. \]

Type 18 is the reflection in a vertical line of type 17; therefore by symmetry,
  \[ w_{17} = w_{18} = 1/4. \]

Consequently, we have

\[
\lambda_3(m) = \frac{a_3 + \sum_{j=1}^{18} w_j}{\text{area}(\sigma_0)} - O(m) = (\frac{4}{3} + \frac{1}{2} + 1 + \frac{1}{10} + \frac{1}{10} + \frac{1}{3} + \frac{1}{3} + \frac{4}{5} + \frac{1}{2})m^2 - O(m) \\
= 5m^2 - O(m).
\]

The values of \(\lambda_i(m)\), for \(i = 3, 4, \ldots, 12\), are summarized in Table 11; for convenience the linear terms are omitted. Since \(m = n/12\), \(\lambda_i\) can be also viewed as a function of \(n\).

<table>
<thead>
<tr>
<th>(i)</th>
<th>(3)</th>
<th>(4)</th>
<th>(5)</th>
<th>(6)</th>
<th>(7)</th>
<th>(8)</th>
<th>(9)</th>
<th>(10)</th>
<th>(11)</th>
<th>(12)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\lambda_i(m))</td>
<td>(5m^2)</td>
<td>(103m^2)</td>
<td>(2m^2)</td>
<td>(13m^2)</td>
<td>(11m^2)</td>
<td>(13m^2)</td>
<td>(4m^2)</td>
<td>(m^2)</td>
<td>(m^2)</td>
<td>(m^2)</td>
</tr>
<tr>
<td>(\text{area}(\sigma_0))</td>
<td>(\frac{4}{3})</td>
<td>(\frac{1}{2})</td>
<td>(1)</td>
<td>(\frac{1}{10})</td>
<td>(\frac{1}{10})</td>
<td>(\frac{1}{3})</td>
<td>(\frac{1}{3})</td>
<td>(\frac{4}{5})</td>
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<td></td>
</tr>
<tr>
<td>(\lambda_i(n))</td>
<td>(\frac{5n^2}{144})</td>
<td>(103n^2)</td>
<td>(\frac{2n^2}{144})</td>
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<td>(\frac{m^2}{144})</td>
<td>(\frac{n^2}{144})</td>
<td>(\frac{n^2}{144})</td>
</tr>
</tbody>
</table>

Table 11: The asymptotic values of \(\lambda_i(m)\) and \(\lambda_i(n)\) for \(i = 3, 4, \ldots, 12\).
The multiplicative factor in Eq. (4) is bounded from below as follows:

\[ F(n) \geq \prod_{i=3}^{12} B_i^{\lambda_i(n)} \geq 2^{\frac{5n^2}{144}} \cdot 8^{\frac{103n^2}{144}} \cdot 62^{\frac{60n^2}{144}} \cdot 908^{\frac{11n^2}{144}} \cdot 24698^{\frac{11n^2}{144}} \]
\[ \cdot 1232944^{\frac{n^2}{144}} \cdot 112018190^{\frac{5n^2}{144}} \cdot 18410581880^{\frac{n^2}{144}} \cdot 5449192389984^{\frac{n^2}{144}} \cdot 2894710651370536^{\frac{n^2}{144}} \cdot 2^{-O(n)}. \]

We prove by induction on \( n \) that \( T(n) \geq 2^{c n^2} - O(n \log n) \) for a suitable constant \( c > 0 \).

It suffices to choose \( c \) (using the values of \( B_i \) for \( i = 3, \ldots, 12 \) in Table 1) so that

\[
\frac{1}{144} \left( 5 + \frac{103}{60} \log 8 + \frac{13}{60} \log 62 + \frac{11}{105} \log 24698 + \frac{13}{105} \log 1232944 \right) + \frac{4}{105} \log 112018190 + \frac{1}{20} \log 18410581880 + \frac{1}{30} \log 5449192389984 \]
\[ + \frac{1}{12} \log 2894710651370536 \geq \frac{11c}{12}. \]

The above inequality holds if we set

\[
c = \frac{1}{132} \left( 5 + \frac{103}{60} \log 8 + \frac{2}{5} \log 62 + \frac{13}{60} \log 908 + \frac{11}{105} \log 24698 \right)
\[ + \frac{13}{105} \log 1232944 + \frac{4}{105} \log 112018190 + \frac{1}{20} \log 18410581880 \]
\[ + \frac{1}{30} \log 5449192389984 + \frac{1}{12} \log 2894710651370536 \geq 2053.2. \]