ON THE AVERAGE COMPLEXITY OF THE $K$-LEVEL

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Abstract. Let $L$ be an arrangement of $n$ lines in the Euclidean plane. The $k$-level of $L$ consists of all vertices $v$ of the arrangement which have exactly $k$ lines of $L$ passing below $v$. The complexity (the maximum size) of the $k$-level in a line arrangement has been widely studied. In 1998 Dey proved an upper bound of $O(n \cdot (k+1)^{1/3})$. Due to the correspondence between lines in the plane and great-circles on the sphere, the asymptotic bounds carry over to arrangements of great-circles on the sphere, where the $k$-level denotes the vertices at distance $k$ to a marked cell, the south pole.

We prove an upper bound of $O((k+1)^2)$ on the expected complexity of the ($\leq k$)-level in great-circle arrangements if the south pole is chosen uniformly at random among all cells.

We also consider arrangements of great $(d-1)$-spheres on the $d$-sphere $S^d$ which are orthogonal to a set of random points on $S^d$. In this model, we prove that the expected complexity of the $k$-level is of order $\Theta((k+1)^{d-1})$.

In both scenarios, our bounds are independent of $n$, showing that the distribution of arrangements under our sampling methods differs significantly from other methods studied in the literature, where the bounds do depend on $n$.

1 Introduction

Let $L$ be an arrangement of $n$ lines in the Euclidean plane. The vertices of $L$ are the intersection points of lines of $L$. Throughout this article we consider arrangements to be simple, i.e., no three lines intersect in a common vertex. Moreover, we assume that no line

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*Supported by ERC StG 757609. S. Felsner and M. Scheucher were supported by DFG Grant FE 340/12-1. R. M. Scheucher was supported by the internal research funding “Post-Doc-Funding” from Technische Universität Berlin. Steiner was supported by DFG-GRK 2434. P. Schnider was supported by the SNSF Project 200021E-171681. P. Valtr was supported by the grant no. 18-1915S of the Czech Science Foundation (GAČR) and by the PRIMUS/17/SCI/3 project of Charles University. This work was initiated at a workshop of the collaborative DACH project Arrangements and Drawings in Schloss St. Martin, Graz.

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is vertical. The $k$-level of $\mathcal{L}$ consists of all vertices $v$ which have exactly $k$ lines of $\mathcal{L}$ below $v$. The $(\leq k)$-level of $\mathcal{L}$ consists of all vertices $v$ which have at most $k$ lines of $\mathcal{L}$ below $v$. We denote the $k$-level by $V_k(\mathcal{L})$ and its size by $f_k(\mathcal{L})$. Moreover, by $f_k(n)$ we denote the maximum of $f_k(\mathcal{L})$ over all arrangements $\mathcal{L}$ of $n$ lines, and by $f(n) = f_{(n-2)/2}(n)$ the maximum size of the middle level.

A $k$-set of a finite point set $P$ in the Euclidean plane is a subset $K$ of $k$ elements of $P$ that can be separated from $P \setminus K$ by a line. Paraboloid duality is a bijection $P \leftrightarrow \mathcal{L}_P$ between point sets and line arrangements (for details on this duality see [O’R94, Chapter 6.5] or [Ed87, Chapter 1.4]). The number of $k$-sets of $P$ equals $|V_{k-1}(\mathcal{L}_P) \cup V_{n-1-k}(\mathcal{L}_P)|$.

In discrete and computational geometry bounds on the number of $k$-sets of a planar point set, or equivalently on the size of $k$-levels of a planar line arrangement have important applications. The complexity of $k$-levels was first studied by Lovász [Lov71] and Erdős et al. [ELSS73]. They bound the size of the $k$-level by $O(n \cdot (k+1)^{1/2})$. Dey [Dey98] used the crossing lemma to improve the bound to $O(n \cdot (k+1)^{1/3})$. In particular, the maximum size $f(n)$ of the middle level is $O(n^{4/3})$. Concerning the lower bound on the complexity, Erdős et al. [ELSS73] gave a construction showing that $f(2n) \geq 2f(n) + cn = \Omega(n \log n)$ and conjectured that $f(n) \geq \Omega(n^{1+\epsilon})$. An alternative $\Omega(n \log n)$-construction was given by Edelsbrunner and Welzl [EW85]. The current best lower bound $f_k(n) \geq n \cdot \Omega(\sqrt{\log k})$ was obtained by Nivasch [Niv08] improving the constant on a bound of the same asymptotic by Tóth [Tót01]. The complexity of the $(\leq k)$-level in arrangements of lines is better understood. Alon and Győri [AG86] prove a tight upper bound of $(k+1)(n-k/2-1)$ for its size. For further information, we recommend the survey by Wagner [Wag08].

1.1 Generalized Zone Theorem

In order to define “zones”, let us introduce the notion of “distances”. For $x$ and $x'$ being a vertex, edge, line, or cell of an arrangement $\mathcal{L}$ of lines in $\mathbb{R}^2$ we let their distance $\text{dist}_{\mathcal{L}}(x, x')$ be the minimum number of lines of $\mathcal{L}$ intersected by the interior of a curve connecting a point of $x$ with a point of $x'$. Pause to note that the $k$-level of $\mathcal{L}$ is precisely the set of vertices which are at distance $k$ to the bottom cell.

The $(\leq j)$-zone $Z_{\leq j}(\ell, \mathcal{L})$ of a line $\ell$ in an arrangement $\mathcal{L}$ is defined as the set of vertices, edges, and cells from $\mathcal{L}$ which have distance at most $j$ from $\ell$. See Figure 1(a) for an illustration.

For arrangements of hyperplanes in $\mathbb{R}^d$ the $(\leq j)$-zone is defined similarly. The classical zone theorem provides bounds for the complexity of the zone $(\leq 0)$-zone of a hyperplane (cf. [ESS91] and [Mat02, Chapter 6.4]). A generalization with bounds for the complexity of the $(\leq j)$-zone appears as an exercise in Matoušek’s book [Mat02, Exercise 6.4.2]. In the proof of Theorem 2 we use a variant of the 2-dimensional case (Proposition 1). For the sake of completeness and to provide explicit constants, we include the proof of Proposition 1 in Section 3.

Proposition 1. Let $\mathcal{L}$ be a simple arrangement of $n$ lines in $\mathbb{R}^2$ and $\ell \in \mathcal{L}$. The $(\leq j)$-zone of $\ell$ contains at most $2e \cdot (j+1)n$ vertices strictly above $\ell$. 


1.2 Arrangements of Great-Circles

Let $\Pi$ be a plane in 3-space which does not contain the origin and let $S^2$ be a sphere in 3-space centered at the origin. The central projection $\Psi_{\Pi}$ yields a bijection between arrangements of great circles on $S^2$ and arrangements of lines in $\Pi$. Figure 1(b) gives an illustration.

The correspondence $\Psi_{\Pi}$ preserves interesting properties, e.g. simplicity of the arrangements. If $\Psi_{\Pi}(C) = L$ and $L$ has no parallel lines, then $\Psi_{\Pi}$ induces a bijection between pairs of antipodal vertices of $C$ and vertices of $L$.

As in the planar case, we define the distance between points $x, y$ of $S^2$ with respect to a great-circle arrangement $C$ as the minimum number of circles of $C$ intersected by the interior of a curve connecting $x$ with $y$. The $k$-level ($\leq k$)-level resp.) of $C$ is the set of all the vertices of $C$ at distance $k$ (distance at most $k$ resp.) from the south pole. The $\leq j$)-zone of a great-circle in $S^2$ is defined similar to the $\leq j$)-zone of a line in $\mathbb{R}^2$.

Let $\Pi_1$ and $\Pi_2$ be two parallel planes in 3-space with the origin between them and let $\Psi_1$ and $\Psi_2$ be the respective central projections. For a great-circle arrangement $C$ we consider $L_1 = \Psi_1(C)$ and $L_2 = \Psi_2(C)$. A vertex $v$ from the $k$-level of $C$ maps to a vertex of the $k$-level in one of $L_1$, $L_2$ and to a vertex of the $(n - k - 2)$-level in the other. Hence, bounds for the maximum size of the $k$-level of line arrangements carry over to the $k$-level of great-circle arrangements except for a multiplicative factor of 2.

The $\leq j$)-zone of a great-circle $C$ in $C$ projects to a $\leq j$)-zone of a line in each of $L_1$ and $L_2$. Hence, the complexity of a $\leq j$)-zone in $C$ is upper bounded by two times the maximum complexity of a $\leq j$)-zone in a line arrangement. Proposition 1 implies that the $\leq j$)-zone of a great-circle $C$ in an arrangement of $n$ great-circles contains at most $4e \cdot (j + 1)n$ vertices in each of the two open hemispheres bounded by $C$. 

\begin{figure}[h]
\begin{center}
\includegraphics[width=\textwidth]{figure1.png}
\end{center}
\caption{(a) The higher order zones of a line $\ell$. (b) The correspondence between great-circles on the unit sphere and lines in a plane. Using the center of the sphere as the center of projection points on the sphere are projected to the points in the plane.}
\end{figure}
1.3 Higher Dimensions

The problem of determining the complexity of the $k$-level admits a natural extension to higher dimensions. We consider arrangements in $\mathbb{R}^d$ of hyperplanes to be simple, meaning that no $d+1$ hyperplanes intersect in a common point. Moreover, we assume that no hyperplane is parallel to the $x_d$-axis. The $k$-level of $A$ consists of all vertices (i.e. intersection points of $d$ hyperplanes) which have exactly $k$ hyperplanes of $A$ below them (with respect to the $d$-th coordinate). We denote the $k$-level by $V_k(A)$ and its size by $f_k(A)$. Moreover, by $f^{(d)}(n)$ we denote the maximum of $f_k(A)$ among all arrangements $A$ of $n$ hyperplanes in $\mathbb{R}^d$.

As in the planar case, there remains a gap between lower and upper bounds;

$$\Omega(n^{\lfloor d/2 \rfloor k^{\lfloor d/2 \rfloor - 1}}) \leq f^{(d)}(n) \leq O(n^{\lfloor d/2 \rfloor k^{\lfloor d/2 \rfloor - c_d}}),$$

here $c_d > 0$ is a small positive constant only depending on $d$. Details and references can be found in Chapter 11 of Matoušek’s book [Mat02]. In dimensions 3 and 4 improved bounds have been established. For example, for $d = 3$, it is known that $f_k^{(3)}(n) \leq O(n(k + 1)^{3/2})$ (see [SST01]). For the middle level in dimension $d \geq 2$ an improved lower bound $f^{(d)}(n) \geq n^{d-1} \cdot e^{\Omega(\sqrt{\log n})}$ is known (see [Tót01] and [Niv08]).

We call the intersection of $\mathbb{S}^d$ with a central hyperplane in $\mathbb{R}^{d+1}$ a great-$(d-1)$-sphere of $\mathbb{S}^d$. Similar to the planar case, arrangements of hyperplanes in $\mathbb{R}^d$ are in correspondence with arrangements of great-$(d-1)$-spheres on the unit sphere $\mathbb{S}^d$ (embedded in $\mathbb{R}^{d+1}$). The terms “distance” and “$k$-level” generalize in a natural way.

2 Our Results

In the first part of this paper we consider arrangements of great-circles on the sphere and investigate the average complexity of the $k$-level when the southpole is chosen uniformly at random among the cells. This question was raised by Barba, Pilz, and Schnider while sharing a pizza [BPS19, Question 4.2].

In Section 4 we prove the following bound on the average complexity.

**Theorem 2.** Let $C$ be a simple arrangement of great-circles. The expected size of the $(\leq k)$-level is at most $16e \cdot (k + 2)^2$ when the southpole is chosen uniformly at random among the cells of $C$.

Note that for $k \geq n/4$ the bound is meaningless, since it exceeds the number of vertices of the arrangement. Our proof works for $k < n/3$ which is needed for Lemma 5. It is remarkable that the bound is independent of the number $n$ of great-circles in the arrangement.

In the second part, we investigate arrangements of randomly chosen great-circles. Here we propose the following model of randomness. On $\mathbb{S}^2$ we have the duality between points and great-circles (each antipodal pair of points defines the normal vector of the plane
containing a great-circle). Since we can choose points uniformly at random from $S^2$, we get random arrangements of great-circles. The duality generalizes to higher dimensions so that we can talk about random arrangements on $S^d$ for a fixed dimension $d \geq 2$. Using the duality between antipodal pairs of points on $S^d$ and great-$(d-1)$-spheres, we prove the following bound on the expected size of the $k$-level in this random model (the proof can be found in Section 5). Again the bound does not depend on the size of the arrangement.

**Theorem 3.** Let $d \geq 2$ be fixed. In an arrangement of $n$ great-$(d-1)$-spheres chosen uniformly at random on the unit sphere $S^d$ (embedded in $\mathbb{R}^{d+1}$), the expected size of the $k$-level is of order $\Theta((k+1)^{d-1})$ for all $k \leq n/2$.

### 3 Proof of Proposition 1

As hinted in Matoušek’s book [Mat02, Exercise 6.4.2], we use the method of Clarkson and Shor [CS89] to prove Proposition 1.

Let $\mathcal{L}$ be an arrangement of $n$ lines in $\mathbb{R}^2$ and let $\ell \in \mathcal{L}$ be a fixed line. For any $j = 0, 1, \ldots, n-1$ denote by $V_{\leq j}$ the set of vertices of $\mathcal{L}$ contained in the $(\leq j)$-zone $Z_{\leq j}(\ell, \mathcal{L})$ of $\ell$ and lying strictly above $\ell$. In other words, $v \in V_{\leq j}$ if there is a simple path $P_v$ in the halfplane $\ell^+$ from $v$ to $\ell$ whose interior has at most $j$ intersections with lines from $\mathcal{L}$.

Let $R$ be a random sample of lines from $\mathcal{L}$ where $\ell \in R$ and each line $\ell' \neq \ell$ independently belongs to $R$ with probability $p := \frac{1}{j+1}$. The probability that a vertex $v \in V_{\leq j}$ is present in the induced subarrangement $\mathcal{L}(R)$ and appears at distance 0 from $\ell$ is at least $(\frac{1}{j+1})^r \cdot (1 - \frac{1}{j+1})^r$, where $0 \leq r \leq j$ denotes the distance of $v$ from $\ell$ in $\mathcal{L}$. Figure 2 gives an illustration. Note that

$$
\left(1 - \frac{1}{j+1}\right)^r \geq \left(1 - \frac{1}{j+1}\right)^j = \left(\frac{j}{j+1}\right)^j = \left(1 + \frac{1}{j}\right)^{-j} \geq 1/e.
$$

![Figure 2: A path $P_v$ witnessing that $v$ belongs to the $(\leq j)$-zone of $\ell$ for all $j \geq 2.$](image-url)

Let $X$ be the number of vertices in the 0-zone of $\ell$ in $\mathcal{L}(R)$ that lie strictly above $\ell$. For the expectation of this random variable we have

$$
\mathbb{E}(X) \geq \frac{1}{e} \left(\frac{1}{j+1}\right)^2 \cdot |V_{\leq j}|.
$$
An inductive argument, as used to show the classical zone theorem (see [GHW13, page 136]), shows there are at most $2n - 3$ vertices lying strictly above $\ell$ in the zone. Hence, we have $X \leq 2 \cdot |R|$ and
\[
\mathbb{E}(X) \leq 2 \cdot \mathbb{E}(|R|) = 2np.
\]
The above inequalities imply
\[
|V_{\leq j}| \leq e \cdot (j + 1)^2 \cdot 2 \cdot n \cdot p = 2 \cdot e \cdot (j + 1) \cdot n.
\]
This concludes the proof of the theorem.

4 Proof of Theorem 2

For the proof of Theorem 2, we fix a great-circle $C$ from $C$ and denote the two closed hemispheres bounded by $C$ on $\mathbb{S}^2$ as $C^+$ and $C^-$. As an intermediate step, we bound the size of the set $F_{\leq k}(C^+)$ of pairs $(F, v)$, where $F$ is a cell of $C^-$ touching $C$ and $v$ is a vertex of $C^+$ whose distance to $F$ is at most $k$. The main ingredient to the proof of the theorem is to show $|F_{\leq k}(C^+)| \leq 8e \cdot (k + 1)^2 n$. We begin with auxiliary considerations.

Consider a family $\mathcal{I}$ of half-intervals in $\mathbb{R}$, it consists of left-intervals of the form $(-\infty, a]$ and right-intervals $[b, \infty)$. A subset $J$ of $k$ half-intervals from $\mathcal{I}$ is a $k$-clique if there is a point $p \in \mathbb{R}$ that lies in all the half-intervals of $J$ but not in any half-interval of $\mathcal{I} \setminus J$. Similarly, a $(\leq k)$-clique is defined as a clique of size at most $k$.

Lemma 4. Any family $\mathcal{I}$ of half-intervals in $\mathbb{R}$ contains at most $2k + 1$ different $(\leq k)$-cliques.

Proof. For $p \in \mathbb{R}$, let $l(p)$ be the number of left-intervals and $r(p)$ the number of right-intervals containing $p$. A point $p$ certifies a $(\leq k)$-clique if and only if $l(p) + r(p) \leq k$. From the monotonicity of the functions $l$ and $r$ it follows that if $(l(p_1), r(p_1)) = (l(p_2), r(p_2))$ for two points $p_1$ and $p_2$, then they are contained in the same sub-interval. Thus, they certify the same clique. In other words, when we move from one sub-interval to its right sub-interval, either $l$ is decreased by 1 or $r$ is increased by 1. We proceed to bound the number of sub-intervals corresponding to $(l, r)$-pairs whose sum is at most $k$.

Let $I_1$ be the leftmost sub-interval such that its $(l, r)$-pair $(l_1, r_1)$ satisfies $l_1 + r_1 \leq k$, and let $I_2$ be the rightmost sub-interval such that its $(l, r)$-pair $(l_2, r_2)$ satisfies $l_2 + r_2 \leq k$. The number of sub-intervals between $I_1$ and $I_2$ (including them) is $l_1 - l_2 + r_2 - r_1 + 1$ because of the monotonicity of $l$- and $r$-values. This number is at most $2k + 1$ because $l_2, r_1 \geq 0$ and $l_1, r_2 \leq k$. Now, the definition of $I_1$ and $I_2$ implies that the number of $(\leq k)$-cliques is most $2k + 1$.

The next lemma is a corresponding result for half-circles on the circle $\mathbb{S}^1$.

Lemma 5. Any family $\mathcal{H}$ of $n$ half-circles in $\mathbb{S}^1$ with $n > 3k$ contains at most $2k + 1$ different $(\leq k)$-cliques.
Proof. For this proof, we embed $S^1$ as the unit-circle in $\mathbb{R}^2$, which is centered at the origin $o$. We consider the set $X$ of all points from $S^1$, which are contained in at most $k$ of the half-circles of $H$, and distinguish the following two cases.

Case 1: The origin $o$ is not contained in the convex hull of $X$. There is a line separating $o$ from $X$ and rotational symmetry allows us to assume that $X$ is contained in the half-plane $\Pi^+ = \{(x,y) \in \mathbb{R}^2; y > 0\}$. For each half-circle $C \in H$, the central projection of $C \cap \Pi^+$ to the line $y = 1$ is a half-interval. Since $(\leq k)$-cliques of $H$ and $(\leq k)$-cliques of the half-intervals are in bijection we get from Lemma 4 that $H$ has at most $2k + 1$ different $(\leq k)$-cliques.

Case 2: The origin $o$ is contained in the convex hull of $X$. By Carathéodory’s theorem, we can find three points $p_1, p_2, p_3$ such that $o$ lies in the convex hull of $p_1, p_2, p_3$. Since each of the $n$ half-circles from $H$ contains at least one of these three points, and each of these three points lies on at most $k$ half-circles, we have $n \leq 3k$, which contradicts the assumption that $n > 3k$. \hfill \Box

For a fixed vertex $v \in C^+ \setminus C$, let $B_{C^+}(v)$ be the set of cells $F$ such that $(F,v) \in \mathcal{F}_{\leq k}(C^+)$, in particular $\text{dist}(F,v) \leq k$.

Claim. $|B_{C^+}(v)| \leq 2k - 1$.

Proof. Consider a great-circle $D \neq C$ from $C$. For a point $x \in C$, we say that $(v,x)$ is $D$-separated if every path from $v$ to $x$ in $C^+$ intersects $D$. The set of all $D$-separated points forms a half-circle $H_D$ on $C$. Let $H$ be the set of these half-circles, i.e., $H = \{H_D : D \in C, D \neq C\}$. See Figure 3.

![Figure 3: An illustration of the cyclic half-circles $H$.](image)

We claim that there is a bijection between $B_{C^+}(v)$ and the $(\leq k - 1)$-cliques in $H$. Indeed, if the intersection of the half-circles of a clique $K$, viewed as a subset of $C$, is $I_K$, then $I_K$ is the interval of $C$ which is reachable from $v$ by crossing the circles corresponding to the half-circles of $K$. If $F$ is a cell from $C^-$ at distance $i \leq k$ from $v$, then $C$ and a subset of $i - 1$ additional circles have to be crossed to reach $v$ from $F$, i.e., there is a $(\leq k - 1)$-clique in $H$ whose intersection is $F \cap C$. The number of $(\leq k - 1)$-cliques in $H$ is at most $2k - 1$ by Lemma 5. \hfill \Box
Claim. $|F_{\leq k}(C^+)| \leq 8e \cdot (k + 1)^2 n$.

Proof. In the case of $k = 0$, vertex $v$ must be one of the $2n - 2$ vertices on $C$ and $F$ is one of the two cells of $C^-$ which is adjacent to $v$. Hence, $|F_{\leq 0}(C^+)| \leq 4n \leq 8e \cdot (k + 1)^2 n$.

Let $k \geq 1$ and note that if $(F, v) \in F_{\leq k}(C^+)$ then $v$ belongs to the $(\leq k - 1)$-zone of $C$ and $F \in B_{C^+}(v)$. As already noted in Section 1.2, the $(\leq k - 1)$-zone of $C$ contains at most $4e \cdot kn$ vertices of $C^+ \setminus C$ and $2n - 2$ vertices on $C$. From the above claim we have $|B_{C^+}(v)| \leq 2k - 1$ for any $v \in C^+ \setminus C$. For the vertices $v$ on $C$, there are only $2k + 2$ cells of $C^-$ touching $C$ with distance at most $k$ to $v$. Hence we conclude that $|F_{\leq k}(C^+)| \leq 4e \cdot kn \cdot (2k - 1) + (2n - 2) \cdot (2k + 2) \leq 8e \cdot (k + 1)^2 n$. \hfill \Box

Since $C$ was chosen arbitrarily among all great-circles from $C$ and $C^+$ was chosen arbitrarily among the two hemispheres of $C$, the upper bound from the above claim holds for any induced hemisphere of $C$. For the union $F_{\leq k}$ of the $F_{\leq k}(C^+)$ over all the $2n$ choices of the hemisphere $C^+$, we have

$$|F_{\leq k}| \leq \sum_{C^+ \text{ hemisphere}} |F_{\leq k}(C^+)| \leq 16e(k + 1)^2 n^2.$$

Proof of Theorem 2. The $(\leq k)$-level with the southpole chosen in cell $F$ consists of the vertices at distance at most $k$ from $F$. Thus, the expected complexity of the $(\leq k)$-level when choosing $F$ uniformly at random equals $|F_{\leq k}|$ divided by the number of cells. Since the number of cells in an arrangement of $n$ great-circles is $2\binom{n}{2} + 2$ and $|F_{\leq k}| \leq 16e(k + 1)^2 n^2$, we can conclude the statement from

$$\frac{16e \cdot (k + 1)^2 \cdot n^2}{2\binom{n}{2} + 2} \leq 16e \cdot (k + 1)^2 \cdot \frac{n}{n - 1} \leq 16e \cdot (k + 2)^2 \cdot \frac{k + 1}{k + 2} \cdot \frac{n}{n - 1} \leq 1.$$

5 Proof of Theorem 3

Let $C$ be a simple arrangement of $n$ great-$(d - 1)$-spheres on the unit sphere $S^d = \{x \in \mathbb{R}^{d+1} : ||x|| = 1\}$ with center $o = (0, \ldots, 0)$ in $\mathbb{R}^{d+1}$. For a vertex $v$ of the arrangement, let $\phi_C(v)$ denote the number of great-$(d - 1)$-spheres of $C$ that are crossed by the geodesic arc from $v$ to the south-pole $s = (0, \ldots, 0, -1)$ of the sphere. The set of vertices $v$ of $C$ with $\phi_C(v) = k$ is denoted $V_k(C)$.

When $C$ is projected to a $d$-dimensional plane $H$ with the origin $o$ as center of projection, we obtain an arrangement $A$ of hyperplanes in $\mathbb{R}^d$. Moreover, if the south pole $s$ is projected to a point “at infinity” of $H$, say to $(0, \ldots, 0, -\infty)$, then, for every point $p$ in $S^d$, the circle in $S^d$ containing the geodesic arc from $p$ to $s$ is projected to the “vertical” line through $p$, i.e., the line $p + (0, \ldots, 0, \lambda)$. The geodesic is projected to one of the two rays starting from $p$ on this line. In particular, all vertices $v$ of $C$ with $\phi_C(v) = k$ are projected to vertices of $A$ either at level $k$ or $n - k - d$. 

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JoCG 11(1), 493–506, 2020
Let $\mathcal{C}$ be an arrangement of randomly chosen great-$(d-1)$-spheres and let $\mathcal{B}$ be a subset of size $d$ in $\mathcal{C}$. Note that with probability 1, the random great-sphere-arrangement is simple, i.e., no great-sphere contains the south-pole and no more than $d$ great-spheres intersect in a common point. Choose $p'$ as one of the two intersection points of the great-$(d-1)$-spheres in $\mathcal{B}$. Now consider the arrangement $\mathcal{C}' = \mathcal{C} - \mathcal{B}$ and note that $(\mathcal{C}', p')$ can be viewed as a random arrangement of great-$(d-1)$-spheres together with a random point on $S^d$. Hence, to estimate the expected size of $V_k(\mathcal{C})$, we can estimate the probability that $\phi_{\mathcal{C}'}(p') = k$. This is the purpose of the following lemma.

**Lemma 6.** Let $\mathcal{C}$ be an arrangement of $n$ great-$(d-1)$-spheres chosen uniformly at random on the unit sphere $S^d$ (embedded in $\mathbb{R}^{d+1}$ and centered at the origin). Let $p$ be an additional point chosen uniformly at random from $S^d$, and let $A$ be the geodesic arc from $p$ to the south pole on $S^d$. For all $k \leq n/2$, the probability $q_k$ that exactly $k$ great-$(d-1)$-spheres from $\mathcal{C}$ intersect $A$ is in $\Theta((k+1)^{d-1}/n^d)$. More precisely, it satisfies

$$2^{d-1}\rho \pi (k+1)^{d-1}(n-k+1)^{d-1}(n+1)^{2d-1} \leq q_k \leq \min\left\{ \frac{\rho \pi n}{n+1}, \frac{\rho \pi (k+1)^{d-1}}{(n+1)^d} \right\},$$

where $a! = a(a+1)\cdots(a+b-1)$ denotes the rising factorial and $\rho = \rho_d = \frac{\text{area}_{d-1}(S^{d-1})}{\text{area}_d(S^d)} = \frac{\Gamma\left(\frac{d+1}{2}\right)}{\pi^{(d+1)/2}}\frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d+1}{2}\right)}$ only depends on the dimension $d$.

**Proof.** Denote by $\phi$ the length of the geodesic arc $A$ on $S^d$ from $p$ to $s$, i.e., $\phi$ is the angle between the two rays emanating from $o$ towards $s$ and $p$. Note that – independent from the dimension $d$ – the three points $o$, $s$, and $p$ lie in a 2-dimensional plane which also contains the geodesic arc $A$.

Point $p$ lies on a $(d-1)$-sphere $C$ of radius $\sin(\phi)$ in the $d$-dimensional hyperplane defined by the equation $x_d = -\cos(\phi)$. Figure 4 gives an illustration for the case $d = 2$, where $C$ is a circle.

**Figure 4:** Illustrating the definitions of $A$, $C$, and $\Pi$ depending on $p$.

The probability that a random great-$(d-1)$-sphere $D$ intersects the arc $A$ defined by the random point $p$ is $\phi/\pi$, since $D$ will intersect the great circle containing $A$ in a random pair of antipodal points. Thus, the probability that $A$ is intersected by exactly $k$
great-$(d-1)$-spheres from the random arrangement $\mathcal{C}$ is

$$q_k = \int_{\phi=0}^{\pi} \frac{\text{area}_{d-1}(S^{d-1}) \sin^{d-1}(\phi)}{\text{area}_d(S^d)} \cdot \left( \binom{n}{k} \frac{(\phi/\pi)^k (1 - \phi/\pi)^{n-k}}{(1 - \phi/\pi)^{n-k}} \right) d\phi.$$

This can be rewritten as

$$q_k = \rho \cdot \left( \binom{n}{k} \int_{\phi=0}^{\pi} \sin^{d-1}(\phi) \cdot \left( \frac{\phi}{\pi} \right)^k (1 - \frac{\phi}{\pi})^{n-k} d\phi, \right.$$

where $\rho = \rho(d) = \frac{\text{area}_{d-1}(S^{d-1})}{\text{area}_d(S^d)} = \frac{\Gamma\left(\frac{d+1}{2}\right)}{\pi^{\frac{d}{2}} \Gamma\left(\frac{d}{2}\right)}$ is a constant only depending on $d$. The latter equation follows from $\text{area}_d(S^d) = 2\pi^{\frac{d+1}{2}} \Gamma\left(\frac{d+1}{2}\right)$, where $\Gamma(x)$ is the Euler gamma function (see e.g. [Wikb]).

In the following we give upper and lower bounds for $q_k$. The Euler beta function $B$ turns out to be the tool to evaluate the integrals:

$$B(a+1, b+1) = \int_{t=0}^{1} t^a (1-t)^b dt = \frac{a! \cdot b!}{(a+b+1)!}.$$

For this identity and more information see for example [Wika].

To show the first upper bound on $q_k$, we bound the integral above as follows: Since $\sin(\phi) \leq 1$ holds for every $\phi \in [0, \pi]$, we have

$$q_k \leq \rho \cdot \left( \binom{n}{k} \int_{\phi=0}^{\pi} \left( \frac{\phi}{\pi} \right)^k (1 - \frac{\phi}{\pi})^{n-k} d\phi = \rho \pi \frac{n!}{k!(n-k)!} \int_{t=0}^{1} t^k (1-t)^{n-k} dt \right).$$

Towards the second upper bound on $q_k$, we use the fact that $\sin(\phi) \leq \phi$ holds for every $\phi \in [0, \pi]$:

$$q_k \leq \rho \pi^{d-1} \frac{n!}{k!(n-k)!} \int_{\phi=0}^{\pi} \left( \frac{\phi}{\pi} \right)^{k+d-1} (1 - \frac{\phi}{\pi})^{n-k} d\phi.$$

$$\frac{n!}{k!(n-k)!} \frac{(k+d-1)! (n-k)!}{(n+d)!} = \rho \pi^d \cdot \frac{(k+1)^{d-1}}{(n+1)^d}.$$
To show the lower bound on \( q_k \), we split the integral in two parts: Since \( \sin(\phi) \geq 2 \cdot \frac{\phi}{\pi} \) holds for every \( \phi \in [0, \pi/2] \) and \( \sin(\phi) \geq 2 \cdot (1 - \frac{\phi}{\pi}) \) holds for every \( \phi \in [\pi/2, \pi] \), we have

\[
q_k \geq 2^{d-1} \rho \left( \binom{n}{k} \right) \left[ \int_{\phi=0}^{\pi/2} (\phi/\pi)^{k+d-1} (1 - \phi/\pi)^{n-k} d\phi + \int_{\phi=\pi/2}^{\pi} (\phi/\pi)^{k} (1 - \phi/\pi)^{n-k+d-1} d\phi \right]
\]

\[
\geq 2^{d-1} \rho \left( \binom{n}{k} \right) \int_{\phi=0}^{\pi} (\phi/\pi)^{k+d-1} (1 - \phi/\pi)^{n-k+d-1} d\phi
\]

\[
= 2^{d-1} \rho \pi \left( \binom{n}{k} \right) \int_{t=0}^{1} t^{k+d-1}(1-t)^{n-k+d-1} dt
\]

\[
= 2^{d-1} \rho \pi \cdot \frac{n!}{k!(n-k)!} \cdot \frac{(k+d-1)!(n-k+d-1)!}{(n+2d-1)!}
\]

\[
= \frac{2^{d-1} \rho \pi (k+1)^{d-1}(n-k+1)^{d-1}}{(n+1)^{2d-1}}.
\]

This completes the proof of Lemma 6. \( \square \)

**Proof of Theorem 3.** Consider an arrangement \( \mathcal{C} \) of \( n + d \) great-\((d-1)\)-spheres \( C_1, \ldots, C_{n+d} \) chosen uniformly and independently at random from \( S^d \). Let \( p \) be a vertex of \( \mathcal{C} \) chosen uniformly at random from the intersection points of \( \mathcal{C} \) (i.e., one of the two points of intersection of \( d \) great-\((d-1)\)-spheres \( C_{i_1}, \ldots, C_{i_d} \) chosen u.a.r. from \( \mathcal{C} \)). Note that \( p \) is a u.a.r. chosen point from \( S^d \).

We now apply Lemma 6 with \( p \) and \( \mathcal{C}_p := \mathcal{C} - \{C_{i_1}, \ldots, C_{i_d}\} \). Point \( p \) is separated from \( s \) by \( k \) great-\((d-1)\)-spheres from \( \mathcal{C}_p \) with probability \( q_k = \Theta(k^{d-1}/n^d) \). Since \( p \) is chosen uniformly at random from \( 2 \binom{n+d}{d} \) vertices of \( \mathcal{C} \), we obtain the desired bound of \( \Theta(k^{d-1}) \) for the number of vertices at distance \( k \) from \( s \). \( \square \)

### 6 Discussion

With Theorem 2 we have shown that the expected size of the \((\leq k)\)-level of a a simple arrangement of great-circles with random south-pole is \( O(k^2) \). With recent work of Goaoc and Welzl [GW20, Prop. 14] this translates to the following dual statement: Let be \( P \) is a set of \( n \) antipodal pairs of points on \( S^2 \). If \( R \) is a labelled affine order type based on \( P \) chosen uniformly at random, then the expected number of \((\leq k)\)-edges of \( R \) is \( O(k^2) \). Here, \( R \) is said to be based on \( P \) if \( R \cup (-R) \) is a labelled copy of \( P \). As a direct consequence of this we obtain that for an uniformly chosen labelled affine order type of size \( n \) the expected number of \((\leq k)\)-edges is \( O(k^2) \). It would be interesting to get a similar result for unlabelled affine order types. Ideas and methods from [GW20] seem to indicate a promising path towards such a result.

Theorem 2 is about arrangements of great-circles. All the elements of the proof, however, carry over to great-pseudocircles whence the result could also be stated for arrangements of great-pseudocircles. Projective arrangements of lines are obtained by antipodal identification from arrangements of great-circles. Hence, if you pick a cell u.a.r.
in a projective arrangement of lines (pseudo-lines) the expected number of vertices at distance at most \( k \) from the cell is as in Theorem 2. If the projection \( \Psi_\Pi \) is used to project an arrangements \( \mathcal{C} \) of great-pseudocircles to an Euclidean arrangement \( \mathcal{L} \) on \( \Pi \) such that the south-poles coincide, then the \( k \)-level of \( \mathcal{C} \) corresponds to the union of the \( k \)- and the \((n - k - 2)\)-level of \( \mathcal{L} \).

With respect to lower bounds we would like to know the answer to:

**Question 1.** Is there a family of arrangements where the expected size of the middle level is superlinear when the southpole is chosen uniformly at random?

Recursive constructions from [EW85] and [ELSS73] show that the \((n/2 - s)\)-level can be in \( \Omega(n \log n) \) for any fixed \( s \). Nevertheless computer experiments suggest that if we choose a random southpole for these examples the expected size of the middle level drops to be linear.

Theorem 3 deals with the average size of the \( k \)-level in arrangements of randomly chosen great-circles. In our model, great-circles are chosen independently and uniformly at random from the sphere. Since point sets, line arrangements, and great-circle arrangements are in strong correspondence, the bound from Theorem 3 also applies to \( k \)-sets in point sets and \( k \)-levels of line arrangements from a specific random distribution.

In the context of Erdős–Szekeres-type problems, several articles made use of point sets which are sampled uniformly at random from a convex shape \( K \) [BF87, Val95, BGAS13, BSV20]. The average size of the convex hull (0-level) is well-studied for such sets of points. If \( K \) is a disk, the convex hull has expected size \( O(n^{1/3}) \), and if \( K \) is a convex polygon with \( m \) sides, the expected size is \( O(m \log n) \) [HP11, PS85, Ray70, RS63].

Bárány and Steiger [BS94] also studied the expected number of \( k \)-sets \((k > 0)\) for point sets that are sampled uniformly at random from a convex shape and other random point sets, such as a spherically symmetric distribution in \( \mathbb{R}^d \). All their bounds depend on \( n \). In particular, the expected size of the convex hull is not constant, which is a substantial contrast to our setting. More recently Bárány et al. [BFG+20] extended the investigations from the uniform distribution on convex sets to arbitrary probability measures. They show a constant bound on the expected size of the convex hull of a random sample of \( n \) points if the probability is ‘concentrated’ around the center of a disk (the notion of concentration used here is delicate, just taking the uniform distribution on a subdisk of smaller radius will not work). The arrangements of our Theorem 3 are obtained from points sampled uniformly at random on the unit sphere. This can also be viewed as sampling under a concentrated probability measure on the plane which is obtained through the central projection. So both of the results are consistent. Goaoc and Welzl [GW20] bound the expected size of the convex hull of a random order type by \( 4 + o(1) \). Last but not least, Edelman [Ede92] showed that the expected number of \( k \)-sets of an allowable sequence is of order \( \Theta(\sqrt{kn}) \).

**References**


